Proof Complexity and Satisfiability Solving for several Combinatorial Principles

Gabriel Istrate
West University of Timișoara and the e-Austria Research Institute
Bd. V. Pârvan 4, cam 047, RO-300223, Timișoara, Romania
gabriel.istrate@acm.org

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1 Introduction

In this talk (based on papers [1, 2] and unpublished work performed with Cosmin Bonciș and Adrian Crăciun, both affiliated with the West University of Timișoara) we discuss the use of statements from topological combinatorics as a source of interesting propositional formulas. Our main example is the Kneser-Lovász theorem and its strengthening due to Schrijver. We first considered these formulas in conjunction with the open problem of separating the Frege and extended Frege propositional proof systems (see [3] for an in-depth presentation)

1.1 Theoretical Results

The Kneser-Lovász theorem is stated as follows:

Proposition 1. Given \( n \geq 2k \geq 1 \) and a function \( c : \binom{n}{k} \rightarrow [n-2k+1] \) there exist two disjoint sets \( A, B \) and a color \( 1 \leq l \leq n-2k+1 \) with \( c(A) = c(B) = l \).

See [4, 5] for readable introductions. It was first established by Lovász, using methods rooted in Algebraic Topology.

We can map this statement to a class (denoted by Kneser\(^n\)\(^k\)) of propositional formulas in the obvious way, by encoding with a boolean variable \( X_{A,i} \) the statement \( c(A) = i \). For \( k = 1 \) we retrieve the wellknown class PHP\(_n\) of formulas encoding the pigeonhole principle.

The following results were proved in [1]:

Theorem 1. For all \( k \geq 1, n \geq 3 \) there exists a variable substitution \( \Phi_k \), \( \Phi_k : \text{Var}(\text{Kneser}_{k+1}^n) \rightarrow \text{Var}(\text{Kneser}_{k}^{n-2}) \) such that \( \Phi_k(\text{Kneser}_{k+1}^n) \) is a formula consisting precisely of the clauses of \( \text{Kneser}_{k}^{n-2} \) (perhaps repeated and in a different order).

Therefore, all known existing proof complexity lower bounds for the pigeonhole formulas extend to the Kneser\(^n\)\(^k\) formulas, and the class Kneser\(^n\)\(^k\) + 1 is ”harder” than Kneser\(^n\)\(^k\).
For $k = 2$ and $k = 3$, the cases where the Kneser-Lovász theorem has combinatorial proofs, we proved:

**Theorem 2.** The following are true:

- (a) The class of formulas $\text{Kneser}_{n}^{2}$ has polynomial size Frege proofs.
- (b) The class of formulas $\text{Kneser}_{n}^{3}$ has polynomial size extended Frege proofs.

We continued the investigation of Kneser formulae in [2], where the following surprising results were proved:

**Theorem 3.** For fixed parameter $k \geq 1$, the propositional translations $\text{Kneser}_{n}^{k}$ of the Kneser-Lovász theorem have polynomial size extended Frege proofs.

**Theorem 4.** For fixed parameter $k \geq 1$, the propositional translations $\text{Kneser}_{n}^{k}$ of the Kneser-Lovász theorem have quasi-polynomial size Frege proofs.

Remarkably, the proof of the previous two theorems do not use techniques from algebraic topology, but instead reduce each case of the Kneser-Lovász theorem with fixed $k \geq 1$ to the verification of a finite number of instances. Therefore each such fixed case has efficient combinatorial proofs.

2 Truncations of combinatorial results

A complementary perspective provided in paper [2] concerned the encoding of a combinatorial principle, the so-called *octahedral Tucker lemma*, a discrete version of the Borsuk-Ulam theorem that is strong enough to prove the Kneser-Lovász theorem. The propositional encoding of this principle is inefficient (leads to exponential-size formulas). However, in [2] we found a weaker but polynomial-size truncation. Therefore, our initial intuitions on the hardness of Kneser formulas may in fact hold for the harder truncated Tucker formulas.

In the talk we will present some complexity-theoretic results on the truncated Tucker formulae, as well as a new (unpublished) second example of the truncation of the Octahedral Tucker lemma that is strong enough to establish another combinatorial result, the so-called *necklace-splitting theorem* due to Noga Alon [6].

3 Experiments: solving Kneser formulae in practice

We will discuss experiments performed using the SAT solver *lingeling* and the Integer Programming solver *GUROBI* in solving various instances of the Schrijver formulas.
References


