Mathematical Foundations for Conceptual Blending

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Part I

Conceptual Blending: from Fauconnier and Turner to Goguen

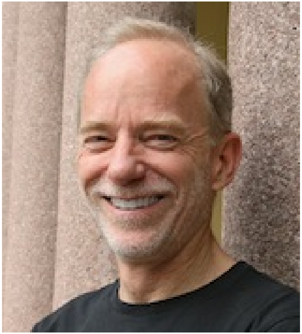
Concepts

- Concepts are basic feature of the functioning of human mind
  1. acquired (through culture),
  2. innate (e.g. as the most subtle sense of “I”)

  (The subject of our study is the former category.)
- Systematic study of concept fabrication within *cognitive sciences*.
- Significance in *computational creativity*, new area of artificial intelligence. Machine fabrication of concepts seen as crucial.

Fauconnier and Turner

Seminal work of *Fauconnier* and *Turner* (cognitive scientists – around year 2000) on systematic fabrication of (new) concepts through so-called *blending*.



Goguen's framework

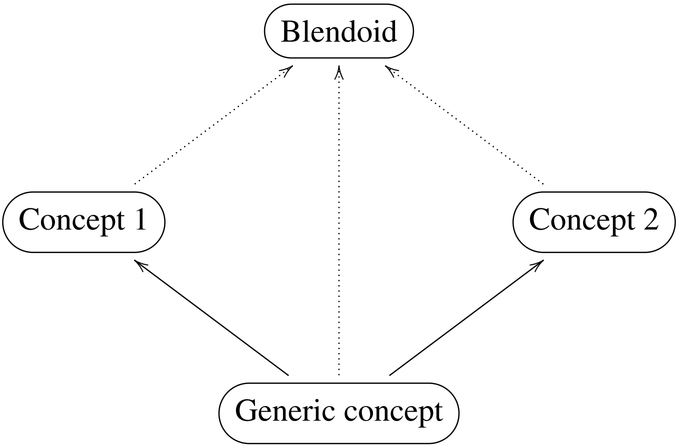
He proposed a general *mathematical* framework for conceptual blending:

- based upon ideas from his own work on *algebraic semiotics*;
- based on *category theory*;
- concepts are represented as *logical theories*;
- he developed several examples using the OBJ formal notation (hence *equational logic* representation of concepts).

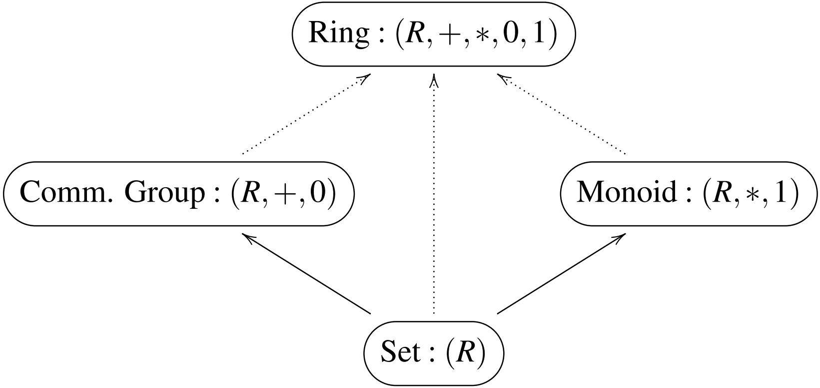
The COINVENT project (2014–2016)

- big European project (FP7);
- aimed to develop a computationally feasible, cognitively-inspired formal model of concept creation, drawing on Fauconnier and Turner's theory of conceptual blending;
- built around Goguen's formal framework;
- concrete applications to mathematics, music, sociology, etc.

Generic scheme for conceptual blending

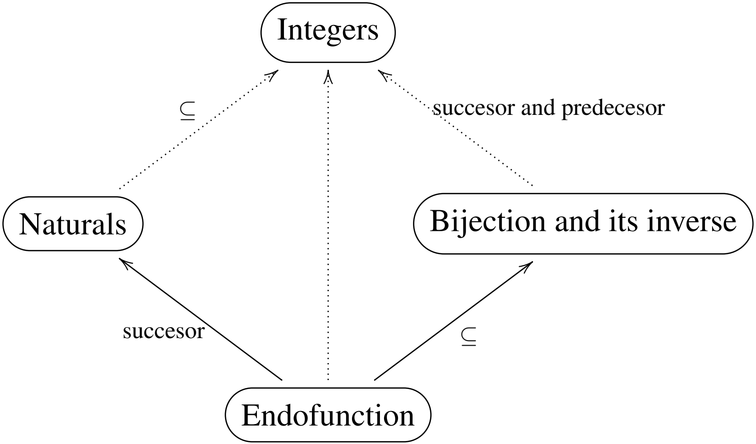


From math: rings as a blending



Note: this is not a *co-limit* of theories as distributivity has to be added.

Another example from math (by Alan Smail)



Partiality sneaks in

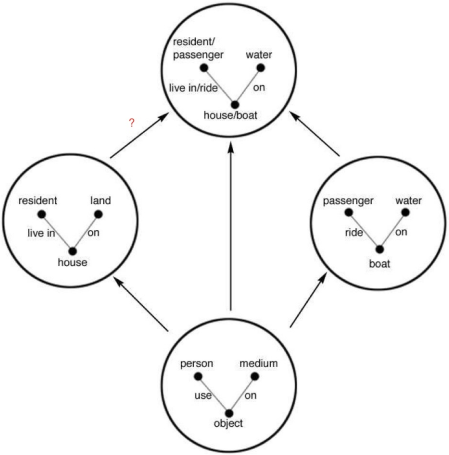
If taking colimit of logical theory morphisms then the axiom $0 \neq \mathrm{succ}(x)$ leads to inconsistency; hence needs to be removed.

Consequently the translation Naturals -> Integers ought to be *partial*.

Conclusion: the arrows in the blending diagram may represent *partial* rather than total translations!

However, I think this is a *non-example* of blending (this will be clarified later on).

Now a classic one: the *houseboat*

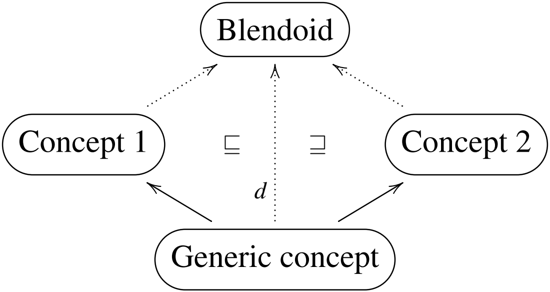


As land cannot be mapped (to water), there is a partiality in the upper-right mapping, and commutativity of the left triangle is lost.

Goguen's $\frac{3}{2}$-categorical approach

The abstract approach to partial mappings requires *order-enriched* category theory, i.e. hom-sets are posets, etc. (a diluted variant of *2-category theory*).

Thus the blending diagram reads as a *lax co-cone* (which means *non-strict* commutativity):



Shortcomings of this approach

Lax co-cones are too many, and apparently no type of $\frac{3}{2}$-co-limit is able to capture adequately the blending diagrams (Goguen proposal of "$\frac{3}{2}$-pushouts" has problems).

But more severely, this has no *proper* semantic dimension given by a model theory (including also a satisfaction relation).

Consequently no way to properly address crucial issues such as *consistency* of blending/merge.

A *refinement* of the $\frac{3}{2}$-categorical approach to an *institution-theoretic* one seems thus unavoidable...

Part II

$\frac{3}{2}$-institutions: refining institution theory with implicit partiality

What is Institution Theory?

- A very general mathematical study of formal logical systems, with emphasis on semantics
- started in the 80's by *Joseph Goguen* and *Rod Burstall*.


What is Institution Theory? (...continued)

- Based upon a mathematical definition for the informal notion of logical system, called *institution*.
- Accommodates not only well established logical systems but also very unconventional ones and moreover it has served and it may serve as a template for defining new ones.
- Approaches logic from a relativistic, non-substantialist perspective, quite different from the common reading of logic.

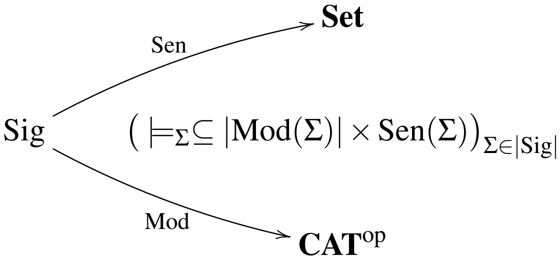
What is Institution Theory? (...continued again)

- Not opposed to the established logic tradition, since it rather includes it from a higher abstraction level.
- Real difference made at the level of methodology, top-down (in the case of institution theory) versus bottom-up (in the case of conventional logic tradition).
- Strong impact in computer science and logic over more than 3 decades. Very big literature.
- Ever growing interest in institution theory by computer scientists and (recently) logicians.

The concept of Institution: the categorical structure

'Logical system' as mathematical object:

$\mathscr{I} = (\mathrm{Sig}, \mathrm{Sen}, \mathrm{Mod}, \models)$:



The concept of Institution: the Satisfaction Condition

$$\begin{array}{ccc} \Sigma & \rho \in \mathrm{Sen}(\Sigma) & \mathrm{Mod}(\varphi)(M') \models_{\Sigma} \rho \\ \varphi \big\downarrow & & \Updownarrow \\ \Sigma' & M' \in |\mathrm{Mod}(\Sigma')| & M' \models_{\Sigma'} \mathrm{Sen}(\varphi)(\rho) \end{array}$$

- Expresses the invariance of truth with respect to change of notation;
- inspired from Barwise-Feferman's abstract model theory.

Application domains

- Foundations for logic-based computing/specification:
  - Sets a standard style for developing new formal specification languages.
  - A great part of modern formal specification theory has been developed at the general level of abstract institutions, including both classical and modern things (heterogeneous, etc.).
  - Exported also to areas such as logic programming, ontologies, etc.
- Redesign of in-depth model theory at the level of abstract institutions:
  - Important difficult new results obtained in model theory,
  - novel approaches have been opened,
  - proper support to powerful unconventional methods (e.g. logic-by-translation), etc.

R. Diaconescu.
Institution-independent Model Theory.
Springer Basel (2008).

Can institutions be used as such as foundations for blending?

NO,

institutions (in their ordinary form) are no good

because they cannot accomodate the emblematic aspect of *partiality* of theory morphisms,

which means

- the sentence translations $\mathrm{Sen}(\varphi)$ ought to be *partial* rather than total,
- the model reducts $\mathrm{Mod}(\varphi)$ ought to be a *blend between functors and relations* rather than functors (since $\mathrm{Mod}(\varphi)(M')$ yields a *set* of models – possibly empty! – rather than a single model), and
- so on...

A $\frac{3}{2}$-dimensional extension of institution theory

- Sig is a $\frac{3}{2}$-category;
- Sen(φ) are partial functions and $\varphi \le \theta$ implies $\mathrm{Sen}(\varphi) \subseteq \mathrm{Sen}(\theta)$;
- Mod(φ) are *lax* functors $\mathrm{Mod}(\Sigma') \to \mathscr{P}\mathrm{Mod}(\Sigma)$:
  - $\mathrm{Mod}(\varphi)(h_1);\mathrm{Mod}(\varphi)(h_2) \subseteq \mathrm{Mod}(\varphi)(h_1;h_2)$;
- if $\varphi \le \theta$ then $\mathrm{Mod}(\theta)(M') \subseteq \mathrm{Mod}(\varphi)(M')$;
- $\mathrm{Mod}(\varphi)(\mathrm{Mod}(\varphi')(M'')) \subseteq \mathrm{Mod}(\varphi;\varphi')(M'')$.

- Satisfaction Condition:

$$M' \models_{\Sigma'} \mathrm{Sen}(\varphi)(\rho) \ \text{ if and only if } \ M \models_{\Sigma} \rho$$

for each $M' \in |\mathrm{Mod}(\Sigma')|$, $M \in |\mathrm{Mod}(\varphi)(M')|$ and $\rho \in \mathrm{dom}(\mathrm{Sen}(\varphi))$.

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$\frac{3}{2}$-institutions:

- a $\frac{3}{2}$-category of signatures Sig,
- a $\frac{3}{2}$-functor $\mathrm{Sen}\colon \mathrm{Sig} \to \mathbf{Pfn}$,
  called the *sentence functor*,
- a lax $\frac{3}{2}$-functor $\mathrm{Mod}\colon (\mathrm{Sig})^{op} \to \frac{3}{2}(\mathbf{CAT}_{\mathscr{P}})$,
  called the *model functor*,
- for each signature $\Sigma \in |\mathrm{Sig}|$ a *satisfaction relation*
  $\models_{\Sigma} \subseteq \ |\mathrm{Mod}(\Sigma)| \times \mathrm{Sen}(\Sigma)$
- such that the above *Satisfaction Condition* holds.

Instances of $\frac{3}{2}$-institutions

- Each ordinary (strict) institution, when Sig regarded as having the trivial $\frac{3}{2}$-structure;
- In general, each concrete institution $\mathscr{I}$ determines a $\frac{3}{2}$-institution $\frac{3}{2}$-$\mathscr{I}$ by considering the 'partial morphisms' over Sig; this process is canonical and can be expressed in precise general mathematical terms by using *inclusion systems*;
- There are however examples that fall outside the scope of $\frac{3}{2}$-$\mathscr{I}$, when more refined partiality is involved, when theories involved, etc.

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Current stage of $\frac{3}{2}$-institution theory

- Basic theory (model amalgamation, theory morphisms, etc.) are already developed (although much more complex than in the case of ordinary institutions).
- Based on those, new foundations for conceptual blending proposed:

  blending is a lax-cocone with model amalgamation.

  (from this perspective the Smail's Integers example is not a blending, but the other examples are accommodated).

Planned developments

- Heterogeneous blending (via Grothendieck $\frac{3}{2}$-institutions).
- More thought on concrete examples.
- Deeper clarification of blending co-cones.
- Implementation in Hets.

THE END