Toric ideals and Gröbner bases

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Toric ideals

Let $A = \{\mathrm{a}_1, \ldots, \mathrm{a}_n\} \subseteq \mathbb{Z}^m$ be a set of vectors in $\mathbb{Q}^m$.
Let $A = [\mathrm{a}_1 \ldots \mathrm{a}_n] \in \mathbb{Z}^{m \times n}$ be an integer matrix with columns a_i. For a vector $\mathrm{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ we let u^+, u^- be the unique vectors in $\mathbb{N}^n$ with disjoint support such that $\mathrm{u} = \mathrm{u}^+ - \mathrm{u}^-$.

Definition

The toric ideal I_A of A is the ideal in $K[x_1, \cdots, x_n]$ generated by all binomials of the form $x^{\mathrm{u}^+} - x^{\mathrm{u}^-}$ where $\mathrm{u} \in \mathrm{Ker}_{\mathbb{Z}}(A)$.

A toric ideal is a binomial ideal.

Binomials in a toric ideal

Toric ideals are binomial ideals.
There are certain sets of binomials that are important:

- Graver basis
- Circuits
- Markov bases
- Indispensable binomials
- reduced Gröbner basis
- universal Gröbner basis

Markov basis

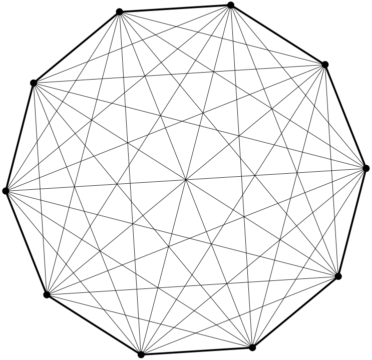
Theorem (Diaconis-Sturmfels 1998)

M is a minimal Markov basis of A if and only if the set $\{x^{\mathrm{u}^+} - x^{\mathrm{u}^-} \,:\, \mathrm{u} \in M\}$ is a minimal generating set of I_A.

Definition

We call a minimal Markov basis of I_A any minimal generating set of I_A.

Markov bases



In the toric ideal of the complete graph on 10 vertices there are

$$3^{210}$$

different minimal Markov bases. Every minimal Markov basis contains 420 elements.

Generic toric ideals

A toric ideal I_A is called generic if it is minimally generated by binomials with full support.

Example

$$A = \begin{pmatrix} 20 & 24 & 25 & 31 \end{pmatrix}$$

$I_A = < x_3^3 - x_1x_2x_4, x_1^4 - x_2x_3x_4, x_4^3 - x_1x_2^2x_3, x_2^4 - x_1^2x_3x_4,$
$x_1^3x_3^2 - x_2^2x_4^2, x_1^2x_2^3 - x_3^2x_4^2, x_1^3x_4^2 - x_2^3x_3^2 > .$

- Every generic toric ideal has a unique minimal Markov basis.
- If the generic toric ideal is not a principal ideal then none of the generators is a circuit.

How many Markov bases exist?

There are two main cases:

1st case. The semigroup $\mathbb{N}A$ is positive, that means $\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n = \{0\}$.

- Every fiber is finite.
- Every minimal Markov basis has the same number of elements.
- There are finitely many different minimal Markov bases.
- The multiset of fibers for which the elements of a minimal Markov basis belong to is an invariant of the toric ideal.
- All minimal Markov bases are subsets of the Graver basis.

2nd case. The semigroup $\mathbb{N}A$ is not positive, that means $\mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^n \neq \{0\}$.

- Every fiber is infinite.
- Different minimal Markov bases may have different number of elements.
- There are infinitely many different minimal Markov bases.
- The multiset of fibers that the elements of a minimal Markov basis belong to is not invariant of the toric ideal.
- There is at least one minimal Markov basis which is a subset of the Graver basis.

Markov basis

Let $A = [1 \ \ -1]$, the simplest example of a matrix such that the semigroup $\mathbb{N}A$ is not positive, since $(1,1) \in \mathrm{Ker}_{\mathbb{Z}}(A) \cap \mathbb{N}^2$.
The Graver basis of A is $\{1 - xy\}$.
The following sets are some of the infinitely many minimal Markov bases:

- $\{1 - xy\}$
- $\{1 - x^2y^2, 1 - x^3y^3\}$
- $\{1 - x^6y^6, 1 - x^{10}y^{10}, 1 - x^{15}y^{15}\}$
- $\{1 - x^2y^2, x - x^2y\}$
- $\{1 - x^5y^5, xy^3 - x^{2014}y^{2016}\}$

H. Charalambous, A. Thoma, M. Vladoiu, *Markov Bases of Lattice Ideals*

Markov basis

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- $\{1 - x^6y^6, 1 - x^{10}y^{10}, 1 - x^{15}y^{15}\}$
- $\{1 - x^2y^2, x - x^2y\}$
- $\{1 - x^5y^5, xy^3 - x^{2014}y^{2016}\}$

H. Charalambous, A. Thoma, M. Vladoiu, *Markov Bases of Lattice Ideals*

Indispensable binomials

Definition

A binomial that belongs (up to sign) to every binomial generating set of the toric ideal I_A is called indispensable.

- All elements in a minimal Markov basis of a generic toric ideal are indispensable.
- None of the elements in any minimal Markov basis of the toric ideal of the complete graph on 10 vertices is indispensable.

Gröbner bases

A Gröbner basis for an ideal $I \subset k[x_1, \ldots, x_n]$ is a set of generators of the ideal I, not necessarily minimal, with good computational properties.

Monomial ideals

Let k be a field and let $k[x_1, \dots, x_n]$ be the polynomial ring in n variables over k.
A monomial is a product $x^{\mathrm{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $\mathrm{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{N}_0^n$.

Definition

An ideal $I \subset k[x_1, \dots, x_n]$ is called a monomial ideal if it is generated by monomials.

Theorem

Every monomial ideal has a finite unique minimal system of monomial generators.

Monomial ideals

Theorem

Let M be a monomial ideal in $k[x_1, \dots, x_n]$ and let $m_1, \dots, m_s$ be the unique minimal system of monomial generators of M. Then

- *the monomial m belongs to the monomial ideal M if and only if there exists an $i \in \{1, \cdots, s\}$ such that $m = m_i q_i$, where q_i is a monomial in $k[x_1, \dots, x_n]$.*
- *the polynomial $f = a_1 x^{\mathrm{u}_1} + a_2 x^{\mathrm{u}_2} + \dots a_r x^{\mathrm{u}_r}$, with each $a_i \neq 0$, belongs to the monomial ideal M if and only if each monomial x^{u_i} belongs to M, where $i \in \{1, \dots, r\}$.*

Monomial orders

By T^n we denote the set of monomials x^a in $k[x_1, \dots, x_n]$.

$$T^n = \{x^a | a \in \mathbb{N}_0^n\}.$$

Definition

By a monomial order on T^n we mean a binary relation $\leq$ on T^n such that

- for every $x^a \in T^n$ we have $x^a \leq x^a$ (reflexive)
- if $x^a \leq x^b$ and $x^b \leq x^a$ then $x^a = x^b$ (antisymmetric)
- if $x^a \leq x^b$ and $x^b \leq x^c$ then $x^a \leq x^c$ (transitive)
- if $x^a, x^b \in T^n$ then $x^a \leq x^b$ or $x^b \leq x^a$ (total order)
- $1 < x^a$ for all $x^a \in T^n$ with $x^a \neq 1$
- If $x^a < x^b$ then $x^a x^c < x^b x^c$ for all $x^c \in T^n$.

Monomial orders

Definition

We say that $x^a < x^b$ if $x^a \leq x^b$ and $x^a \neq x^b$.

If $n = 1$ then there is a unique monomial order (on T^1).
$1 < x_1$ therefore $x_1 < x_1^2$ therefore $x_1^2 < x_1^3 \cdots$
Thus

$$1 < x_1 < x_1^2 < x_1^3 < \cdots .$$

Lexicographic monomial order

Definition (Lexicographic order with $x_1 > x_2 > \cdots > x_n$)

We define $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} > x^b = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if and only if there exists an $i \in \{1, 2, \cdots, n\}$ such that

$$a_1 = b_1$$

$$\cdots$$

$$a_{i-1} = b_{i-1}$$

$$a_i > b_i.$$

There are $n!$ different Lexicographic monomial orders.
We denote the Lexicographic monomial order $>$ by

$$>_{lex} \ .$$

Degree Lexicographic monomial order

Definition (Degree Lexicographic order with $x_1 > x_2 > \cdots > x_n$)

We define $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} > x^b = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if and only if $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$ or $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$ and there exists an $i \in \{1, 2, \cdots, n\}$ such that

$$a_1 = b_1$$

$$\dots$$

$$a_{i-1} = b_{i-1}$$

$$a_i > b_i.$$

There are $n!$ different Degree lexicographic monomial orders. We denote the Degree lexicographic monomial order $>$ by

$$>_{deglex} \; .$$

Degree Lexicographic monomial order

Definition (Degree reverse lexicographic order with $x_1 > x_2 > \cdots > x_n$)

We define $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} > x^b = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if and only if $a_1 + a_2 + \cdots + a_n > b_1 + b_2 + \cdots + b_n$ or $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$ and there exists an $i \in \{1, 2, \cdots, n\}$ such that

$$a_n = b_n$$

$$\ldots$$

$$a_{i+1} = b_{i+1}$$

$$a_i < b_i.$$

There are n! different Degree reverse lexicographic monomial orders. We denote the Degree reverse lexicographic monomial order $>$ by

$$>_{degrevlex} \, .$$

Monomial order

Example

In the polynomial ring $k[x_1, x_2, x_3]$ with $x_1 > x_2 > x_3$ for the monomials $x_1^2, x_1x_2x_3$ and x_2^3 we have

- $x_1^2 >_{lex} x_1x_2x_3 >_{lex} x_2^3$
- $x_1x_2x_3 >_{deglex} x_2^3 >_{deglex} x_1^2$
- $x_2^3 >_{degrevlex} x_1x_2x_3 >_{degrevlex} x_1^2$

Monomial order

Sometimes we can define a monomial order using a matrix $U \in \mathbb{R}^{m \times n}$. We define $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} > x^b = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if and only if the first (from the top) nonzero coordinate of

$$U(a-b)^t$$

is positive.

Lexicographic order with $x_1 > x_2 > \cdots > x_n$

The lexicographic monomial order with $x_1 > x_2 > \cdots > x_n$ can be defined by the identity $n \times n$ matrix,

$$I_{n\times n} = (\delta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

while any lexicographic monomial order can be defined by a permutation matrix

$$(\delta_{i\sigma(j)}).$$

Degree Lexicographic order with $x_1 > x_2 > \cdots > x_n$

The degree lexicographic monomial order with $x_1 > x_2 > \cdots > x_n$ can be defined by the $n \times n$ matrix,

$$D = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

while any other degree lexicographic monomial order can be defined by a matrix obtained by a permutation of the last n rows of D.

Degree Reverse Lexicographic order with $x_1 > x_2 > \cdots > x_n$

The degree reverse lexicographic monomial order with $x_1 > x_2 > \cdots > x_n$ can be defined by the $n \times n$ matrix,

$$R = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

while any other degree reverse lexicographic monomial order can be defined by a matrix obtained by a permutation of the last n rows of R.

Monomial orders

If $n \geq 2$ then there are infinitely many monomial orders on T^n. For example for $n = 2$ there are infinitely many monomial orders on T^2 defined by the matrices

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

or

$$B = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}$$

where $a, b \in \mathbb{R}_{\geq 0}$.
Note that all are distinct except if $ab = 1$ and $a \notin \mathbb{Q}_{\geq 0}$.

Monomial orders defined by matrices

Theorem (Robbiano)

A matrix $U \in \mathbb{R}^{m\times n}$ defines a monomial order if

- $ker(U) \cap \mathbb{N}_0^n = \{(0, 0, \cdots, 0)\}$
- *the first nonzero coordinate in every column of U is positive.*

Every monomial order can be defined by an appropriate matrix.

Initial monomial

Let $>$ be a monomial order on T^n. Let f be a nonzero polynomial in $k[x_1, \ldots, x_n]$. We may write

$$f = a_1 x^{\mathbf{u}_1} + a_2 x^{\mathbf{u}_2} + \cdots + a_r x^{\mathbf{u}_r},$$

where $a_i \neq 0$ and $x^{\mathbf{u}_1} > x^{\mathbf{u}_2} > \cdots > x^{\mathbf{u}_r}$.

Definition

For $f \neq 0$ in $k[x_1, \ldots, x_n]$, we define the initial monomial of f to be $in_<(f) = x^{\mathbf{u}_1}$. The coefficient a_1 is called the initial coefficient of f and is denoted by c_f. For a subset S of $k[x_1, \ldots, x_n]$ we define the initial monomial ideal of S to be the monomial ideal $in_<(S) = \langle in_<(f) | f \in S \rangle$.

Gröbner bases

Definition

A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal I is called Gröbner basis for I if and only if for all nonzero $f \in I$ there exists $i \in \{1, \ldots, t\}$ such that $in_<(g_i)$ divides $in_<(f)$.

Theorem

A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal I is a Gröbner basis for I if and only if

$$in_<(G) = in_<(I).$$

Gröbner bases

Let $<$ be a monomial order on $k[x_1, \ldots, x_n]$ and let $I \subset k[x_1, \ldots, x_n]$ be an ideal.

Definition

The monomials which do not belong to $in_<(I)$ are called standard monomials.

Example

Let I be the ideal $< x_1^2 - x_2^3, x_2^2 - x_3^3, x_3^2 - x_4^3 >$ of the polynomial ring $k[x_1, x_2, x_3, x_4]$ with the lexicographic monomial order with $x_1 > x_2 > x_3 > x_4$. Then $in <_{lex} (I) =< x_1^2, x_2^2, x_3^2 >$ therefore $\{x_1^2 - x_2^3, x_2^2 - x_3^3, x_3^2 - x_4^3\}$ is a Gröbner basis for I.
The standard monomials are of the form $x_4^i, x_1x_4^i, x_2x_4^i, x_3x_4^i, x_1x_2x_4^i, x_1x_3x_4^i, x_2x_3x_4^i, x_1x_2x_3x_4^i$ for some $i \in \mathbb{N}_0$.

Division

Let $>$ be a monomial order on $k[x_1, \ldots, x_n]$.

Definition

Given polynomials f, g, h in $k[x_1, \ldots, x_n]$ with $g \neq 0$, we say that f reduces to h modulo g, and we write $f \to_g h$, if and only if $in_<(g)$ divides a nonzero term X of f and

$$h = f - \frac{X}{c_g in_<(g)} g.$$

Example

Let $f = x_1^4 x_3 + 2x_1^2 x_2^2 - x_3^5$ and $g = x_1^2 x_2 - x_3$ in $\mathbb{Q}[x_1, x_2, x_3]$ with the lexicographic monomial order with $x_1 > x_2 > x_3$. Then $in_<(g) = x_1^2 x_2$ divides the term $X = 2x_1^2 x_2^2$ and $h =$

$$f - \frac{X}{c_g in_<(g)} g = x_1^4 x_3 + 2x_1^2 x_2^2 - x_3^5 - \frac{2x_1^2 x_2^2}{x_1^2 x_2}(x_1^2 x_2 - x_3) = x_1^4 x_3 + 2x_2 x_3 - x_3^5.$$

Division

Let $>$ be a monomial order in $k[x_1, \ldots, x_n]$.

Definition

Given polynomials $f, f_1, \cdots, f_s, h$ in $k[x_1, \ldots, x_n]$ with $f_i \neq 0$. We say that f reduces to h modulo $F = \{f_1, f_2, \cdots, f_s\}$, and we write

$$f \to_F h$$

if and only if there exists a sequence of indices $i_1, \cdots, i_t$ such that

$$f \to_{f_{i_1}} h_1 \to_{f_{i_2}} h_2 \to \cdots \to_{f_{i_t}} h.$$

Division

Let $>$ be a monomial order on $k[x_1, \ldots, x_n]$.

Definition

A polynomial r is called reduced with respect to a set of non-zero polynomials $F = \{f_1, f_2, \cdots, f_s\}$ if

- $r = 0$ or
- no term of r is a multiple of any $in_<(f_i)$.

Definition

If $f \rightarrow_F r$ and r is reduced with respect to F then r is called a remainder for f modulo F.

Remainder

Remark

The remainder of a polynomial f modulo a set of non-zero polynomials may not be unique.

Example

Let $f = x_1x_2x_3 + 2x_1$ and $F = \{f_1 = x_1x_2 - 1, f_2 = x_2x_3 - x_1\}$ in $\mathbb{Q}[x_1, x_2, x_3]$ with the degree lexicographic monomial order with $x_1 > x_2 > x_3$. Then $f \rightarrow_{f_1} 2x_1 + x_3$ and $f \rightarrow_{f_2} x_1^2 + 2x_1$.
Note that both $2x_1 + x_3$ and $x_1^2 + 2x_1$ are reduced with respect to F, thus both are remainders for f modulo F.

Gröbner bases

Theorem

Let I be a non-zero ideal in $k[x_1, \ldots, x_n]$. The set of non-zero polynomials $G = \{g_1, g_2, \cdots, g_t\} \subset I$ is a Gröbner basis for I if and only if the remainder of any polynomial $f \in k[x_1, \ldots, x_n]$ by G is unique.

Theorem

Let I be a non-zero ideal in $k[x_1, \ldots, x_n]$. The set of non-zero polynomials $G = \{g_1, g_2, \cdots, g_t\} \subset I$ is a Gröbner basis for I if and only if the remainder of any polynomial $f \in I$ by G is zero.

The remainder of any polynomial modulo a Gröbner basis is a linear combination of standard monomials.

S-polynomials

Definition (Buchberger)

Let f, g be two non-zero polynomials in $k[x_1, \ldots, x_n]$. Let $L = LCM(in_<(f), in_<(g))$. The polynomial

$$S(f,g) = \frac{L}{c_f in_<(f)} f - \frac{L}{c_g in_<(g)} g$$

is called the S-polynomial of f and g.

Example

Let $f = 3x^2yz - y^3z^3, g = xy^2 + z^2$ in the polynomial ring $\mathbb{Q}[x, y, z]$ with the lexicographic monomial order with $x > y > z$. Then $L = LCM(in_<(f), in_<(g)) = LCM(x^2yz, xy^2) = x^2y^2z$ and

$$S(f,g) = \frac{x^2y^2z}{3x^2yz} f - \frac{x^2y^2z}{xy^2} g = -xz^3 - \frac{y^4z^3}{3}.$$

Gröbner bases

Remark

The S-polynomial of f and g belongs to the ideal generated by f, g.

Theorem

Let I be a non-zero ideal in $k[x_1, \dots, x_n]$. The set of non-zero polynomials $G = \{g_1, g_2, \cdots, g_t\} \subset I$ is a Gröbner basis for I if and only if the remainder of any polynomial $f \in I$ by G is zero.

Theorem (Buchberger)

Let I be a non-zero ideal in $K[x_1, \dots, x_n]$. The set of non-zero polynomials $G = \{g_1, g_2, \cdots, g_t\} \subset I$ is a Gröbner basis for $I =< g_1, g_2, \cdots, g_t >$ if and only if $S(f, g) \rightarrow_G 0$.

Buchberger's Algorithm

- INPUT: $F = \{f_1, f_2, \cdots, f_t\}$ a set of non-zero polynomials of $K[x_1, \ldots, x_n]$
- OUTPUT: $G = \{g_1, g_2, \cdots, g_s\}$ a Gröbner basis for $I =< f_1, f_2, \cdots, f_t >$.
- SET: $G := F$, $S = \{S(f_i, f_j) | f_i \neq f_j \in G\}$
- WHILE $S \neq \emptyset$ DO
  Choose any $S(f, g) \in S$
  set $S : S - \{S(f, g)\}$
  $S(f, g) \rightarrow_G h$, where h is the remainder modulo G
- IF $h \neq 0$ THEN
  $S := S \cup \{S(u, h) | \text{for all} u \in G\}$
  $G := G \cup \{h\}$.

Buchberger's Algorithm

Let $I =< x^2y + z, xz + y >$ be an ideal in $\mathbb{R}[x, y, z]$. Let $<_{deglex}$ be the degree lexicographic monomial order in $\mathbb{R}[x, y, z]$ with $x > y > z$.

- Set $G_0 = \{g_1 = x^2y + z, g_2 = xz + y\}$ and $S_0 = \{S(g_1, g_2)\}$
- $S_0 \neq \emptyset$. Reduce $S(g_1, g_2)$ with respect to G_0: $S(g_1, g_2) = \frac{x^2yz}{x^2y}(x^2y + z) - \frac{x^2yz}{xz}(xz + y) = -xy^2 + z^2 \to_{G_0} -xy^2 + z^2 \neq 0$
- Set $G_1 = \{g_1, g_2, g_3 = -xy^2 + z^2\}$ and $S_1 = (S_0 - \{S(g_1, g_2)\}) \cup \{S(g_1, g_3), S(g_2, g_3)\}$.
- $S_1 \neq \emptyset$. Reduce $S(g_1, g_3)$ with respect to G_1: $S(g_1, g_3) = \frac{x^2y^2}{x^2y}(x^2y + z) - \frac{x^2y^2}{-xy^2}(-xy^2 + z^2) = xz^2 + yz \to_{G_1} 0$
- Set $G_2 = G_1$ and $S_2 = S_1 - \{S(g_1, g_2)\} = \{S(g_2, g_3)\}$.
- $S_2 \neq \emptyset$. Reduce $S(g_2, g_3)$ with respect to G_2: $S(g_2, g_3) = \frac{xy^2z}{xz}(xz + y) - \frac{xy^2z}{-xy^2}(-xy^2 + z^2) = y^3 + z^3 \to_{G_2} y^3 + z^3 \neq 0$
- Set $G_3 = \{g_1, g_2, g_3, g_4 = y^3 + z^3\}$ and $S_3 = (S_2 - \{S(g_2, g_3)\}) \cup \{S(g_1, g_4), S(g_2, g_4), S(g_3, g_4)\} = \{S(g_1, g_4), S(g_2, g_4), S(g_3, g_4)\}$.

Buchberger's Algorithm

Let $I =< x^2y + z, xz + y >$ be an ideal in $\mathbb{R}[x, y, z]$. Let $<_{deglex}$ be the degree lexicographic monomial order in $\mathbb{R}[x, y, z]$ with $x > y > z$.

- Set $G_0 = \{g_1 = x^2y + z, g_2 = xz + y\}$ and $S_0 = \{S(g_1, g_2)\}$
- $S_0 \neq \emptyset$. Reduce $S(g_1, g_2)$ with respect to G_0: $S(g_1, g_2) = \frac{x^2yz}{x^2y}(x^2y + z) - \frac{x^2yz}{xz}(xz + y) = -xy^2 + z^2 \to_{G_0} -xy^2 + z^2 \neq 0$
- Set $G_1 = \{g_1, g_2, g_3 = -xy^2 + z^2\}$ and $S_1 = (S_0 - \{S(g_1, g_2)\}) \cup \{S(g_1, g_3), S(g_2, g_3)\}$.
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Buchberger's Algorithm

$$S(g_1, g_4) \to_{G_3} 0,$$
$$S(g_2, g_4) \to_{G_3} 0,$$
$$S(g_3, g_4) \to_{G_3} 0.$$

So after three more steps $S = \emptyset$ and therefore

$$\{g_1, g_2, g_3, g_4\}$$

is a Gröbner basis for I.

Gröbner bases

For any nonzero ideal I and for any monomial order there exist Gröbner bases for I. Actually there exist infinitely many.

Definition

A Gröbner basis $G = \{g_1, \dots, g_t\}$ is called a reduced Gröbner basis for I if

- the initial coefficient of g_i is equal to 1 for all $i \in \{1, \dots, t\}$ and
- no monomial in g_i is divisible by any $in_<(g_j)$ for any $j \neq i$.

Theorem

Let $<$ be a monomial order on $k[x_1, \dots, x_n]$ and I_A a toric ideal. Then $\{x^{u_1{}^+} - x^{u_1{}^-}, x^{u_2{}^+} - x^{u_2{}^-}, \cdots, x^{u_s{}^+} - x^{u_s{}^-}\}$ is reduced Gröbner basis with respect to the monomial order $<$ if and only if $x^{u_1{}^+}, x^{u_2{}^+}, \cdots, x^{u_s{}^+}$ are the minimal monomial generators of $in_<(I_A)$ and $x^{u_1{}^-}, x^{u_2{}^-}, \cdots, x^{u_s{}^-}$ are standard monomials.

Reduced Gröbner bases

Theorem

(Buchberger) Let $<$ be a monomial order on $k[x_1, \dots, x_n]$ and I a nonzero ideal. Then I has a unique reduced Gröbner basis with respect to the monomial order $<$.

Elimination order

We consider two sets of variables $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$. Let $<_x$ be any monomial order on the x variables and let $<_y$ any monomial order on the y variables. We can define a new monomial order:

Definition

Let x^a, x^b be monomials in the x variables and y^c, y^d be monomials in the y variables. We define

$$x^a y^c < x^b y^d$$

if and only if $x^a <_x x^b$ or $x^a = x^b$ and $y^c <_y y^d$.
The new monomial order is called an elimination order with the x variables larger than the y variables.

If the $<_x$ monomial order is defined by a matrix A and the $<_y$ monomial order is defined by a matrix B then the elimination order is defined by the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Elimination

Theorem

Let I be a nonzero ideal of $K[x_1, \cdots, x_n, y_1, \cdots, y_m]$ and let $<$ be an elimination order with the x variables larger than the y variables. Let $G = \{g_1, g_2, \cdots, g_t\}$ be a Gröbner basis for I. Then $G \cap K[y_1, \cdots, y_m]$ is a Gröbner basis for the ideal $I \cap K[y_1, \cdots, y_m]$.

Universal Gröbner bases

Although $k[x_1, \ldots, x_n]$, for $n \geq 2$ has infinitely many different monomial orders for a fixed nonzero ideal I there exist finitely many different reduced Gröbner bases for I.

Definition

The universal Gröbner basis of an ideal I is the union of all reduced Gröbner bases $G_<$ of the ideal I as $<$ runs over all monomial orders and is denoted by $UGB(I)$.

The universal Gröbner basis is a finite subset of I and it is a Gröbner basis for I with respect to all monomial orders simultaneously.

Theorem

(V. Weispfenning and N. Schwartz) Universal Gröbner basis exists for every ideal in $k[x_1, \ldots, x_n]$.

Gröbner bases of toric ideals

- Toric ideals are binomial ideals
- Let $f = x^{u^+} - x^{u^-}, g = x^{v^+} - x^{v^-}$ be two non-zero binomials in $k[x_1, \dots, x_n]$ with $x^{v^+} > x^{v^-}$ and such that $x^{v^+} | x^{u^+}$. Then the remainder of the division is zero or a binomial.

$$f \to_g h = (x^{u^+} - x^{u^-}) - \frac{x^{u^+}}{x^{v^+}}(x^{v^+} - x^{v^-}) = \frac{x^{u^+}}{x^{v^+}} x^{v^-} - x^{u^-}.$$

- Let $f = x^{u^+} - x^{u^-}, g = x^{v^+} - x^{v^-}$ be two non-zero binomials in $k[x_1, \dots, x_n]$ with $x^{u^+} > x^{u^-}, x^{v^+} > x^{v^-}$. Let $L = LCM(x^{u^+}, x^{v^+})$. The polynomial

$$S(f,g) = \frac{L}{x^{u^+}}(x^{u^+} - x^{u^-}) - \frac{L}{x^{v^+}}(x^{v^+} - x^{v^-}) = \frac{L}{x^{v^+}} x^{v^-} - \frac{L}{x^{u^+}} x^{u^-}$$

is the S-polynomial of f and g and it is binomial.

- Any reduced Gröbner basis of a toric ideal consists of binomials.

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Any reduced Gröbner basis of a toric ideal consists of binomials.
What kind of binomials?

Theorem (B. Sturmfels)

For any toric ideal I_A we have that the Universal Gröbner basis is a subset of the Graver basis.

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Proof.

Suppose that there exists a binomial $x^{u^+} - x^{u^-}$ in the Universal Gröbner basis which does not belong to the Graver. Then

1. there exists a monomial order $<$ such that $x^{u^+} - x^{u^-}$ is in the reduced Gröbner basis with respect to the monomial order $>$ and
2. there exists a non-zero $x^{v^+} - x^{v^-} \in I_A$, with $x^{v^+} - x^{v^-} \neq x^{u^+} - x^{u^-}$ such that $x^{v^+} | x^{u^+}$ and $x^{v^-} | x^{u^-}$.

The first condition means that x^{u^+} is a minimal generator of $in_<(I_A)$ and x^{u^-} is a standard monomial.

For $x^{v^+} - x^{v^-} \in I_A$ there are two cases:

1. $x^{v^+} > x^{v^-}$ implies $x^{v^+} \in in_<(I_A)$ and divides one of the minimal generators of $in_<(I_A)$, the x^{u^+}. Therefore $x^{v^+} = x^{u^+}$. But then $(x^{v^+} - x^{v^-}) - (x^{u^+} - x^{u^-}) = x^{u^-} - x^{v^-} \in I_A$ is non-zero and $x^{u^-} > x^{v^-}$ (since $x^{v^-} | x^{u^-}$). Therefore $x^{u^-} \in in_<(I_A)$. A contradiction since x^{u^-} is a standard monomial.
2. $x^{v^-} > x^{v^+}$ then $x^{v^-} \in in_<(I_A)$ and divides a standard monomial, the x^{u^-}. Contradiction.

□

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$\square$