Bouquet Algebra of Toric ideals

Apostolos Thoma

Department of Mathematics
University of Ioannina

EMS Summer School on Multigraded Algebra and Applications
Moieciu, Romania

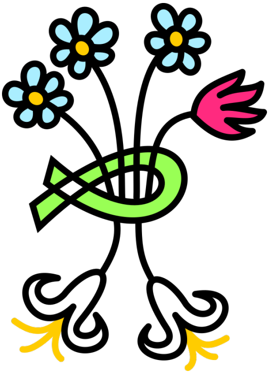
This is joint work with Sonja Petrović and Marius Vladoiu

Bouquet Algebra

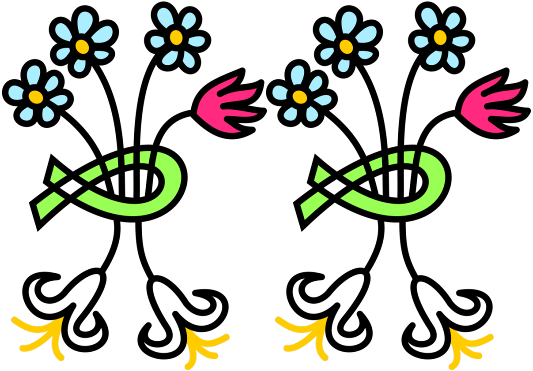
Bouquets are useful for:

- allowing classification-type results
- understanding the complexity of toric ideals of hypergraphs
- reducing certain problems about toric ideals to problems of toric ideals of hypergraphs
- and providing a way to construct families of examples of toric ideals with various interesting properties (robust, generic, unimodular, . . .)

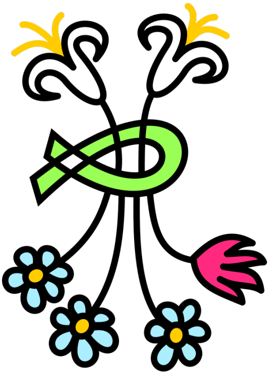
Bouquet algebra



Bouquet algebra



Bouquet algebra



Bouquet Algebra

Example

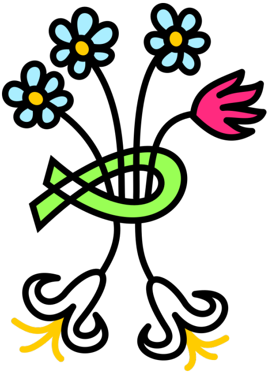
Let A be the integer matrix

$$\begin{pmatrix} 0 & 3 & 0 & 4 & 5 \\ 1 & 3 & 3 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \end{pmatrix},$$

then the elements in the Graver basis of A are:

$$\begin{pmatrix} 3 & 1 & -2 & -2 & 1 \\ 6 & 2 & -4 & 1 & -2 \\ -3 & -1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 5 & -4 \\ 9 & 3 & -6 & -1 & -1 \\ 12 & 4 & -8 & -3 & 0 \\ 15 & 5 & -10 & 0 & -3 \end{pmatrix}$$

Bouquet algebra



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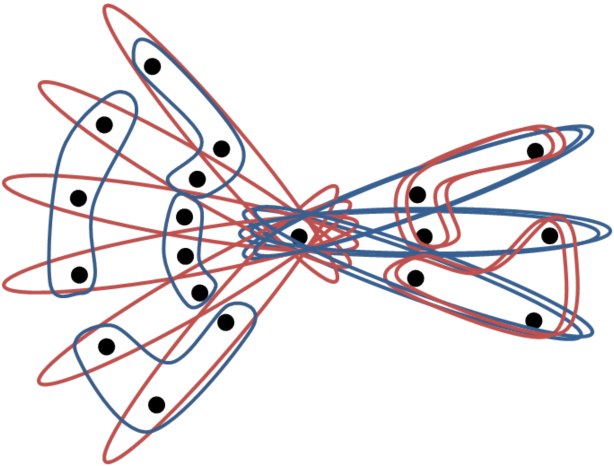
Bouquet algebra



Bouquets

There are two ways to define bouquets:

- using matroid theory or
- using Gale transforms

Bouquet Algebra

The support of a vector $\mathbf{u} \in \mathbb{Z}^n$ is the set $\text{supp}(\mathbf{u}) = \{i | u_i \neq 0\} \subset \{1, \ldots, n\}$.

Definition

- A *co-vector* is any vector of the form $(\mathbf{u} \cdot \mathbf{a}_1, \ldots, \mathbf{u} \cdot \mathbf{a}_n)$.
- A *co-circuit* of A is any non-zero co-vector of minimal support.
- A co-circuit with support of cardinality one is called a *co-loop*.
- We call the vector $\mathbf{a}_i$ *free* if $\{i\}$ is the support of a co-loop.

If $\mathbf{a}_i$ is a free vector then i is not contained in the support of any minimal generator of the toric ideal I_A, or any element in the Graver basis.

Bouquet Graph

Definition

- Let E_A be the set consisting of elements of the form $\{\mathbf{a}_i, \mathbf{a}_j\}$ such that there exists a co-vector $\mathbf{c}_{ij}$ with support $\{i, j\}$.
- Let E_A^+ the subset of E_A where the co-vector is a co-circuit and the signs of the two nonzero coordinates of $\mathbf{c}_{ij}$ are distinct.
- Let E_A^- the subset of E_A where the co-vector is a co-circuit and the signs of the two nonzero coordinates of $\mathbf{c}_{ij}$ are the same.
- Let E_A^0 the subset of E_A where the co-vector is not a co-circuit. This implies that both $\mathbf{a}_i$ and $\mathbf{a}_j$ are free vectors.

By definition, these three sets E_A^+, E_A^-, E_A^0 partition E_A.

Bouquet Graph

Definition

The bouquet graph G_A *of* I_A *is the graph whose vertex set is* $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *and edge set* E_A. *The* bouquets *of A are the connected components of* G_A.

It follows from the co-circuit axioms of oriented matroids that each bouquet of A is a clique in G_A.

Definition

- If there are free vectors in A they form one bouquet with all edges in E_A^0, which we call the *free bouquet* of G_A.
- A non-free bouquet is called *mixed* if it contains at least an edge from E_A^-.
- A non-free bouquet is called *non-mixed* if it is either an isolated vertex or all of its edges are from E_A^+.

Example

Let A be the integer matrix

$$\begin{pmatrix} 0 & 3 & 0 & 4 & 5 \\ 1 & 3 & 3 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \end{pmatrix},$$

Then $a_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$, $a_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ and $a_5 = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$.

Bouquets

Example

Take $u_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ then u_1 defines the co-circuit:

$$(\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}) = (0, 6, 3, 0, 0)$$

therefore $\{a_2, a_3\} \in E_A^-$.

Bouquets

Example

Similarly there are two more co-circuits: the $(2,0,3,0,0)$ defined by $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and the $(-1,3,0,0,0)$ defined by $u_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ therefore $\{a_1, a_3\} \in E_A^-$ and $\{a_1, a_2\} \in E_A^+$.

Bouquet graph

Example

Thus the bouquet graph G_A has five vertices a_1, a_2, a_3, a_4, a_5 and three edges, $\{a_1, a_2\}, \{a_2, a_3\}$ and $\{a_1, a_3\}$.
$E_A^+ = \{\{a_1, a_2\}\}$ and $E_A^- = \{\{a_1, a_3\}, \{a_2, a_3\}\}$.
Therefore there are three bouquets. One of them is mixed: $\{a_1, a_2, a_3\}$, and two non-mixed, $\{a_4\}, \{a_5\}$.

Gale transforms

Denote by r the $\dim_{\mathbb{Q}} \mathbb{Q}(A)$. Fix a basis $G_1, G_2, \ldots, G_{n-r} \in \mathbb{Z}^n$ for the lattice $\mathrm{Ker}_{\mathbb{Z}}(A)$, and denote by G the $n \times (n-r)$ matrix with columns $G_1, \ldots, G_{n-r}$.

Definition

Define $G(A) = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ to be the set of ordered rows of G. The set $G(A)$ is called the *Gale transform* of A, while the vector $G(\mathbf{a}_i) := \mathbf{b}_i$ is called the Gale transform of $\mathbf{a}_i$ for any i.

Bouquet Algebra

Definition

We call a vector a_i free if $G(\mathbf{a}_i) = 0$.

Definition

Let E_A be the set consisting of elements of the form $\{\mathbf{a}_i, \mathbf{a}_j\}$ such that

- both $\{\mathbf{a}_i, \mathbf{a}_j\}$ are free or
- $\mathbf{a}_i, \mathbf{a}_j$ are not free and $G(\mathbf{a}_i) = \lambda G(\mathbf{a}_j)$ for some $\lambda \neq 0$.

Gale transforms

Definition

- The edge $\{\mathbf{a}_i, \mathbf{a}_j\}$ belongs to E_A^+ if and only if $\mathbf{a}_i, \mathbf{a}_j$ are not free and $G(\mathbf{a}_i) = \lambda G(\mathbf{a}_j)$ for some $\lambda > 0$.
- The edge $\{\mathbf{a}_i, \mathbf{a}_j\}$ belongs to E_A^- if and only if $\mathbf{a}_i, \mathbf{a}_j$ are not free and $G(\mathbf{a}_i) = \lambda G(\mathbf{a}_j)$ for some $\lambda < 0$.
- The edge $\{\mathbf{a}_i, \mathbf{a}_j\}$ belongs to E_A^0 if and only if both $\mathbf{a}_i, \mathbf{a}_j$ are free vectors.

Definition

The bouquet graph G_A *of* I_A *is the graph whose vertex set is* $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ *and edge set* E_A*. The* bouquets *of* A *are the connected components of* G_A*.*

Example

Let A be the integer matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

with columns $\{\mathbf{a}_1, \ldots, \mathbf{a}_7\}$.

Example

A basis for $\mathrm{Ker}_{\mathbb{Z}}(A)$ is given by $(1, -1, 0, 0, -1, 1, 0)$ and $(1, 0, -1, -1, 0, 1, 0)$, and thus the Gale transform $G(A)$ of A consists of the following seven vectors: $G(\mathbf{a}_1) = (1, 1)$, $G(\mathbf{a}_2) = (-1, 0)$, $G(\mathbf{a}_3) = (0, -1)$, $G(\mathbf{a}_4) = (0, -1)$, $G(\mathbf{a}_5) = (-1, 0)$, $G(\mathbf{a}_6) = (1, 1)$ and $G(\mathbf{a}_7) = (0, 0)$. Hence $\mathbf{a}_7$ is a free vector. So, the graph G_A has the vertex set $\{\mathbf{a}_1, \ldots, \mathbf{a}_7\}$. The edges of G_A are $\{\mathbf{a}_1, \mathbf{a}_6\}$, $\{\mathbf{a}_2, \mathbf{a}_5\}$ and $\{\mathbf{a}_3, \mathbf{a}_4\}$. Therefore G_A has three bouquets each consisting of a single edge and one additional free bouquet consisting of the isolated vertex $\mathbf{a}_7$. Moreover, since $G(\mathbf{a}_1) = G(\mathbf{a}_6)$, $G(\mathbf{a}_2) = G(\mathbf{a}_5)$ and $G(\mathbf{a}_3) = G(\mathbf{a}_4)$, provides $\{\mathbf{a}_1, \mathbf{a}_6\}, \{\mathbf{a}_2, \mathbf{a}_5\}, \{\mathbf{a}_3, \mathbf{a}_4\} \in E_A^+$. In this case, all of the non-free bouquets are non-mixed.

Subbouquet

Definition

A *subbouquet* of G_A is an induced subgraph of G_A, which is a clique on its set of vertices.

Note that a bouquet is a maximal subbouquet.

Definition

We will say that A has a *subbouquet decomposition* if there exists a family of subbouquets, say $B_1, \dots, B_t$, such that they are pairwise vertex disjoint and their union of vertices equals $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. A subbouquet decomposition always exists if we consider, for example, the subbouquet decomposition induced by all of the bouquets.

bouquet-index-encoding vector

For each bouquet we are define a vector $\mathbf{c}_B$ as follows:

- If the bouquet B is free then we set $\mathbf{c}_B \in \mathbb{Z}^n$ to be any nonzero vector such that $\mathrm{supp}(\mathbf{c}_B) = \{i : \mathbf{a}_i \in B\}$ and with the property that the first nonzero coordinate is positive.
- For a non-free bouquet B of A, consider the Gale transforms of the elements in B. All the elements are nonzero and linearly dependent, therefore there exists a nonzero coordinate j in all of them. Let $g_j = \gcd(G(\mathbf{a}_i)_j | \ \mathbf{a}_i \in B)$ and fix the smallest integer i_0 such that $\mathbf{a}_{i_0} \in B$. Let $\mathbf{c}_B$ be the vector in $\mathbb{Z}^n$ whose i-th coordinate is 0 if $\mathbf{a}_i \notin B$, and is $\varepsilon_{i_0 j} G(\mathbf{a}_i)_j / g_j$ if $\mathbf{a}_i \in B$, where $\varepsilon_{i_0 j}$ represents the sign of the integer $G(\mathbf{a}_{i_0})_j$.

Thus the $\mathrm{supp}(\mathbf{c}_B) = \{i : \mathbf{a}_i \in B\}$. Note that the choice of i_0 implies that the first nonzero coordinate of $\mathbf{c}_B$ is positive. Since each $\mathbf{a}_i$ belongs to exactly one bouquet the supports of the vectors $\mathbf{c}_{B_i}$ are pairwise disjoint. In addition, $\cup_i \mathrm{supp}(\mathbf{c}_{B_i}) = [n]$.

Whether the bouquet B is mixed or not can be read off from the vector $\mathbf{c}_B$ as follows.

Lemma

Suppose B is a non-free bouquet of A. Then B is a mixed bouquet if and only if the vector $\mathbf{c}_B$ has a negative and a positive coordinate.

It follows from Lemma and the definition of the vector $\mathbf{c}_B$ that if the non-free bouquet B is non-mixed, then all nonzero coordinates of $\mathbf{c}_B$ are positive.

The following vector now encodes the set of dependencies from the Gale transform, and, therefore, also all of the essential bouquet information about B.

Definition

Let B be a bouquet of A and define

$$\mathbf{a}_B := \sum_{i=1}^{n} (c_B)_i \mathbf{a}_i,$$

where $(c_B)_i$ denotes the i-th coordinate of the vector $\mathbf{c}_B$. The set of all vectors $\mathbf{a}_B$ corresponding to all bouquets of A will be denoted by A_B, and thought of as a matrix with columns $\mathbf{a}_B$.

Theorem

Suppose the bouquets of $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ *are* $B_1, \dots, B_s$, *and define* $A_B = [\mathbf{a}_{B_1}, \dots, \mathbf{a}_{B_s}]$. *There is a bijective correspondence between the elements of* $\mathrm{Ker}_{\mathbb{Z}}(A)$ *and the elements of* $\mathrm{Ker}_{\mathbb{Z}}(A_B)$ *given by the map* $\mathbf{u} \mapsto B(\mathbf{u})$, *where for* $\mathbf{u} = (u_1, \dots, u_s) \in \mathrm{Ker}_{\mathbb{Z}}(A_B)$

$$B(\mathbf{u}) := \mathbf{c}_{B_1} u_1 + \cdots + \mathbf{c}_{B_s} u_s.$$

It is an immediate consequence of the Theorem that the ideals I_A and I_{A_B} have the same codimension, as the kernels of the two matrices have the same rank.

Example

Let A be the integer matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

with columns $\{\mathbf{a}_1, \ldots, \mathbf{a}_7\}$. It has three bouquets, B_1, B_2, B_3, with two vertices each: $\{\mathbf{a}_1, \mathbf{a}_6\}$, $\{\mathbf{a}_2, \mathbf{a}_5\}$ and $\{\mathbf{a}_3, \mathbf{a}_4\}$, and the isolated vertex $\mathbf{a}_7$ as the free bouquet B_4.

Example

Let us compute the nonzero vectors $\mathbf{c}_{B_1}, \mathbf{c}_{B_2}, \mathbf{c}_{B_3}, \mathbf{c}_{B_4} \in \mathbb{Z}^7$. For $\mathbf{c}_{B_1}$, j can be chosen either 1 or 2, while $i_0 = 1$. Fix $j = 1$. Thus $g_1 = \gcd(G(\mathbf{a}_1)_1, G(\mathbf{a}_6)_1) = 1$ and the nonzero coordinates of $\mathbf{c}_{B_1}$ are

$$(c_{B_1})_1 = \varepsilon_{11}\frac{G(\mathbf{a}_1)_1}{g_1} = 1, \ (c_{B_1})_6 = \varepsilon_{11}\frac{G(\mathbf{a}_6)_1}{g_1} = 1.$$

Hence $\mathbf{c}_{B_1} = (1,0,0,0,0,1,0)$ and $\mathbf{a}_{B_1} = \mathbf{a}_1 + \mathbf{a}_6 = (1,1,1,1,0)$. Similarly it can be computed that $\mathbf{c}_{B_2} = (0,1,0,0,1,0,0)$, $\mathbf{a}_{B_2} = \mathbf{a}_2 + \mathbf{a}_5 = (1,1,1,1,0)$, $\mathbf{c}_{B_3} = (0,0,1,1,0,0,0)$ and $\mathbf{a}_{B_3} = \mathbf{a}_3 + \mathbf{a}_4 = (1,1,1,1,0)$, while by definition $\mathbf{c}_{B_4} = (0,0,0,0,0,0,1)$ and $\mathbf{a}_{B_4} = \mathbf{a}_7 = (0,0,0,0,3)$.

Example

Then $A_B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$, and consequently $\mathrm{Ker}_{\mathbb{Z}}(A_B) = \{(\alpha+\beta, -\alpha, -\beta, 0) |\ \alpha, \beta \in \mathbb{Z}\}$. This encodes the vector

$$B((\alpha+\beta, -\alpha, -\beta, 0)) = (\alpha+\beta)\mathbf{c}_{B_1} - \alpha\mathbf{c}_{B_2} - \beta\mathbf{c}_{B_3} =$$

$$= (\alpha+\beta, -\alpha, -\beta, -\beta, -\alpha, \alpha+\beta, 0),$$

which is a generic element of $\mathrm{Ker}_{\mathbb{Z}}(A)$.

Bouquet Algebra

Theorem

Suppose the bouquets of $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$ are $B_1, \ldots, B_s$, and define $A_B = [\mathbf{a}_{B_1}, \ldots, \mathbf{a}_{B_s}]$. Then there is a bijective correspondence between the Graver basis of A_B and the Graver basis of A, and a similar bijective correspondence holds between the sets of circuits. Explicitly:

$$Graver(A) = \{B(\mathbf{u})|\ \mathbf{u} \in Graver(A_B)\}$$

$$\text{and } \ \mathcal{C}(A) = \{B(\mathbf{u})|\ \mathbf{u} \in \mathcal{C}(A_B)\}.$$

This correspondence in general does not offer any relation between Markov bases, indispensable binomials and universal Gröbner bases of A and A_B.

Bouquet Algebra

Example

$Graver(A_B) = \mathcal{C}(A_B) = \{(1,-1,0,0),(0,1,-1,0),(1,0,-1,0)\}$, while $Graver(A) = \mathcal{C}(A)$ and consists of the vectors $(1,-1,0,0,-1,1,0),(0,1,-1,-1,1,0,0),(1,0,-1,-1,0,1,0)$, since

$$B((u_1,u_2,u_3,u_4)) = u_1\mathbf{c}_{B_1} + u_2\mathbf{c}_{B_2} + u_3\mathbf{c}_{B_3} + u_4\mathbf{c}_{B_4} =$$

$$= (u_1,u_2,u_3,u_3,u_2,u_1,u_4).$$

The one-to-one correspondence for the two sets holds in the order indicated. For example,

$$B((0,1,-1,0)) = \mathbf{c}_{B_2} - \mathbf{c}_{B_3} = (0,1,-1,-1,1,0,0).$$

Subbouquet

The two previous Theorems are true even if we replace bouquets with proper subbouquets which form a subbouquet decomposition of A.

Generalized Lawrence matrices

Next we will see a construction of a natural inverse procedure of the one given previously.
Given an arbitrary set of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_s$ and vectors $\mathbf{c}_1, \ldots, \mathbf{c}_s$ that can act as bouquet-index-encoding vectors the following result constructs a toric ideal I_A whose s subbouquets are encoded by the given vectors.

Generalized Lawrence matrices

Recall that an integral vector $\mathbf{a} \in \mathbb{Z}^m$ is primitive if the greatest common divisor of all its coordinates is 1.

Theorem

Let $\{\mathbf{a}_1, \dots, \mathbf{a}_s\} \subset \mathbb{Z}^m$ be an arbitrary set of vectors. Let $\mathbf{c}_1, \dots, \mathbf{c}_s$ be any set of primitive vectors, with $\mathbf{c}_i \in \mathbb{Z}^{m_i}$ for some $m_i \geq 1$, each having full support and a positive first coordinate. Define $p = m + \sum_{i=1}^{s}(m_i - 1)$ and $q = \sum_{i=1}^{s} m_i$. Then, there exists a matrix $A \in \mathbb{Z}^{p \times q}$ with a subbouquet decomposition, $B_1, \dots, B_s$, such that the i^{th} subbouquet is encoded by the following vectors: $\mathbf{a}_{B_i} = (\mathbf{a}_i, \mathbf{0}, \dots, \mathbf{0}) \in \mathbb{Z}^p$ and $\mathbf{c}_{B_i} = (\mathbf{0}, \dots, \mathbf{c}_i, \dots, \mathbf{0}) \in \mathbb{Z}^q$, where the support of $\mathbf{c}_{B_i}$ is precisely in the i^{th} block of $\mathbb{Z}^q$ of size m_i.

Generalized Lawrence matrices

We construct A explicitly. For each $i = 1, \ldots, s$, let $\mathbf{c}_i = (c_{i1}, \ldots, c_{im_i}) \in \mathbb{Z}^{m_i}$ and define

$$C_i = \begin{pmatrix} -c_{i2} & c_{i1} & & & \\ -c_{i3} & & c_{i1} & & \\ & & & \ddots & \\ -c_{im_i} & & & & c_{i1} \end{pmatrix} \in \mathbb{Z}^{(m_i-1)\times m_i}.$$

Generalized Lawrence matrices

Primitivity of each $\mathbf{c}_i$ implies that there exist integers $\lambda_{i1}, \ldots, \lambda_{im_i}$ such that $1 = \lambda_{i1} c_{i1} + \cdots + \lambda_{im_i} c_{im_i}$. Fix a choice of $\lambda_{i1}, \ldots, \lambda_{im_i}$, and define the matrices $A_i = [\lambda_{i1}\mathbf{a}_i, \ldots, \lambda_{im_i}\mathbf{a}_i] \in \mathbb{Z}^{m \times m_i}$. The desired matrix A is then the following block matrix:

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_s \end{pmatrix} \in \mathbb{Z}^{p \times q},$$

where $p = m + (m_1 - 1) + \cdots + (m_s - 1)$ and $q = m_1 + \cdots + m_s$.

Generalized Lawrence matrices

We will call the matrix

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_s \end{pmatrix} \in \mathbb{Z}^{p \times q},$$

the generalized Lawrence matrix, since in particular one can recover, after a column permutation, the classical second Lawrence lifting.

Generalized Lawrence matrices

Corollary

For any integer matrix A there exists a generalized Lawrence matrix A' such that $I_A = I_{A'}$, up to permutation of column indices.

Application for Unimodular matrices

The most important property of the previous Theorem is that it can be used to provide infinite classes of examples having a particular property. For example, if we want to construct infinitely many unimodular matrices we proceed as follows: let $D = [\mathbf{a}_1, \ldots, \mathbf{a}_s]$ be any unimodular matrix (for example the incidence matrix of a bipartite graph) and choose arbitrary $\mathbf{c}_1, \ldots, \mathbf{c}_s$ with entries -1 or 1, and the first nonzero coordinate being 1. Then for the generalized Lawrence matrix A such that its subbouquet ideal equals I_D we can prove that A is also unimodular.

Generalized Lawrence matrices

Similarly, from an arbitrary unimodular matrix D, using any set of primitive vectors $\mathbf{c}_1, \ldots, \mathbf{c}_s$ and such that at least one has one coordinate in absolute value larger than 1, we can construct infinitely many generalized Lawrence matrices that are not unimodular, but have the set of circuits equal to the universal Gröbner basis and the Graver basis.

Generalized Lawrence matrices

- Given an arbitrary graph G whose connected components are cliques then there exists a matrix A such that the bouquet graph G_A of A is precisely G.
- Given a graph G whose connected components are cliques, along with $+$ and $-$ signs associated to each edge of G according to the sign rules that in any triangle if two edges are positive then the third is positive and if two edges are negative then the third is positive, then there exists a matrix A whose bouquet graph and structure are encoded by G.

Next we are going to study the two extreme special cases:

- all the the bouquets of A are non-mixed or free.
- the bouquets of A are mixed or free.

Stable toric ideals

Definition

The toric ideal I_A is called *stable* if all of the bouquets of A are non-mixed or free.

Theorem

Let I_A be a stable toric ideal. Then the bijective correspondence between the elements of $\mathrm{Ker}_{\mathbb{Z}}(A)$ and $\mathrm{Ker}_{\mathbb{Z}}(A_B)$ given by $\mathbf{u} \mapsto B(\mathbf{u})$, is preserved when we restrict to any of the following sets: Graver basis, circuits, indispensable binomials, minimal Markov bases, reduced Gröbner bases (universal Gröbner basis).

Application to generic toric ideals

Recall that a toric ideal I_A is called *generic* if it is minimally generated by binomials with full support. The following result states that in the case of stable toric ideals genericity is preserved when passing from A to A_B and conversely.

Theorem

Let I_A be a stable toric ideal. Then I_A is a generic toric ideal if and only if I_{A_B} is a generic toric ideal.

This theorem provides a way to construct an infinite class of generic toric ideals starting from an arbitrary example of a generic toric ideal.

Generic toric ideals

Example

$$A = \begin{pmatrix} 20 & 24 & 25 & 31 \end{pmatrix}$$

$I_A = < x_3^3 - x_1x_2x_4, x_1^4 - x_2x_3x_4, x_4^3 - x_1x_2^2x_3, x_2^4 - x_1^2x_3x_4,$
$x_1^3x_3^2 - x_2^2x_4^2, x_1^2x_2^3 - x_3^2x_4^2, x_1^3x_4^2 - x_2^3x_3^2 >$.

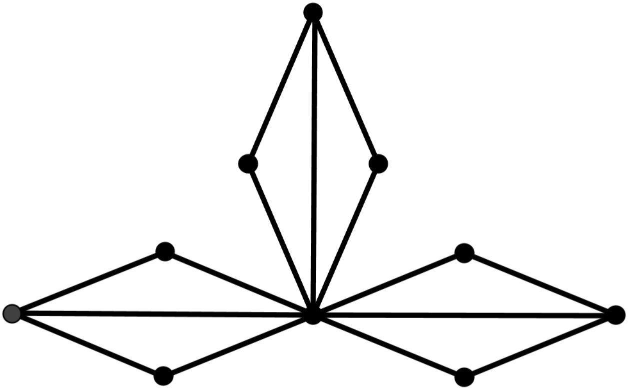
all mixed bouquets

Theorem

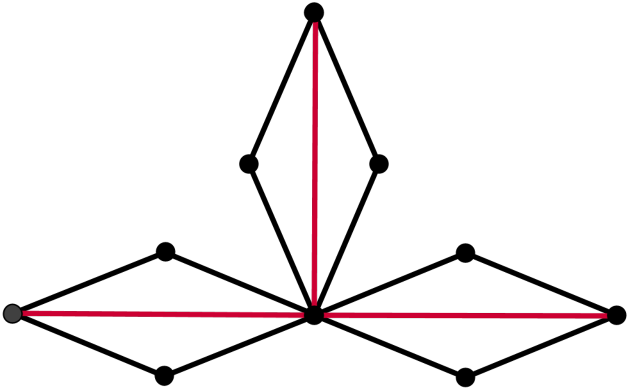
Suppose that every non-free bouquet of A is mixed. Then the following sets of binomials coincide:

1. *the Graver basis of A,*
2. *the universal Gröbner basis of A,*
3. *any reduced Gröbner basis of A,*
4. *any minimal Markov basis of A,*
5. *the indispensable elements of A.*

Bouquet algebra



Bouquet algebra



How complicated are the toric ideals of hypergraphs?

Hypergraphs

Theorem

Given any integer matrix A, there exists a hypergraph $H = (V, E)$ such that:

1. *There is a bijective correspondence, $u \mapsto B(u)$, between $\mathrm{Ker}_{\mathbb{Z}}(A)$ and $\mathrm{Ker}_{\mathbb{Z}}(H)$, and between the Graver basis of (A) and the Graver basis of (H),*
2. *For every $u \in Graver(A)$ we have $\deg(x^{u^+} - x^{u^-}) \leq \deg(x^{B(u^+)} - x^{B(u^-)})$,*
3. *The following four sets of I_H coincide:*
   - *the Graver basis of H,*
   - *the universal Gröbner basis of H,*
   - *any reduced Gröbner basis of H,*
   - *any minimal Markov basis of H.*

Hypergraphs encode all positively graded toric ideals

Theorem

Let I_D be an arbitrary positively graded nonzero toric ideal. Then there exists a hypergraph H such that there is a one-to-one correspondence between the Graver basis, the universal Gröbner basis reduced Gröbner bases, minimal Markov bases, indispensable binomials, and circuits of I_D and I_H.

Hypergraphs encode all positively graded toric ideals

Let

$$D = \begin{pmatrix} 1 & 3 & 2 & 0 & 1 \\ 3 & 2 & 1 & 3 & 2 \\ 3 & 0 & 2 & 2 & 1 \end{pmatrix} \in \mathbb{N}^{3\times 5}.$$

$$
\left(\begin{array}{cccc|cccc|ccc|cccc|ccc}
1&0&0&0&0&1&1&1&1&1&0&0&0&0&0&1&0&0\\
1&0&0&0&1&0&1&1&1&1&0&0&0&0&0&1&0&0\\
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\hline
1&1&1&0&0&0&0&0&1&1&0&1&1&0&0&1&0&0
\end{array}\right)
$$

Problems about arbitrary positively graded toric ideals related to equality of different bases can be reduced to problems about toric ideals of hypergraphs.

Robust toric ideals

Definition

Robust toric ideals are those for which the universal Gröbner basis is a minimal Markov basis.

Theorem (Boocher, Brown, Duff, Lyman, Murayama, Nesky, Schaefer)

Let I_A a robust toric ideal of a graph. Then the Graver basis is a minimal Markov basis.

Let I_A a robust toric ideal. Then the Graver basis is a minimal Markov basis?

Thank you