Special Topics in Many-Valued Logics

Part I - Preliminaries on Łukasiewicz logic

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Special Topics in Computer Science

January, 2018

Many-valued logics

- non-classical
- truth functional
- truth value $\{0, 1\} \subseteq W \subseteq [0, 1]$
- truth is a matter of degree (P. Hájek)
- originated from the papers of Łukasiewicz and Post in the twenties

Logic	Classical logic
Truth values	$L_2 = \{0, 1\}$

Łukasiewicz logic

Logic	∞-valued Łukasiewicz logic $\mathcal{L}_\infty$	n-valued Łukasiewicz logic $\mathcal{L}_n$
Truth values	$[0,1]$	$L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$

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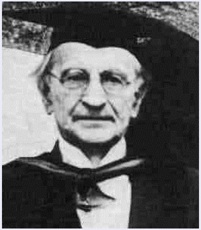
"I should like to state only one thing, namely that determinism is not a view better justified than indeterminism."

J. Łukasiewicz, On determinism, 1946



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Łukasiewicz connectives on L_n and $[0,1]$

$$\neg p := 1 - p \text{ and } p \to q := \min(1 - p + q, 1)$$

Derived connectives

$p \vee q = \max(p,q) = (p \to q) \to q$,
$p \wedge q = \min(p,q) = \neg(\neg p \vee \neg q)$

∞-valued Łukasiewicz logic $\mathcal{L}_\infty$:

Connectives: $\{\neg, \rightarrow\}$

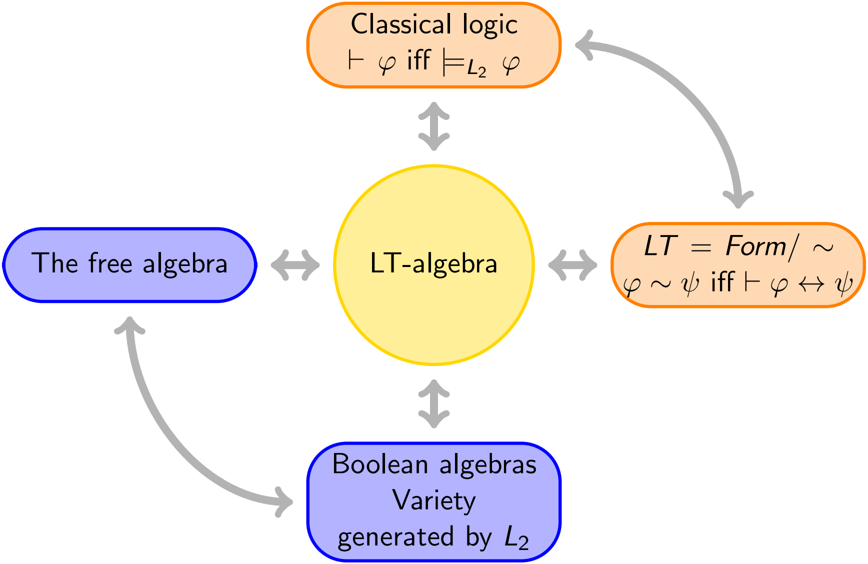
Axioms

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$;
2. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$;
3. $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$;
4. $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$.

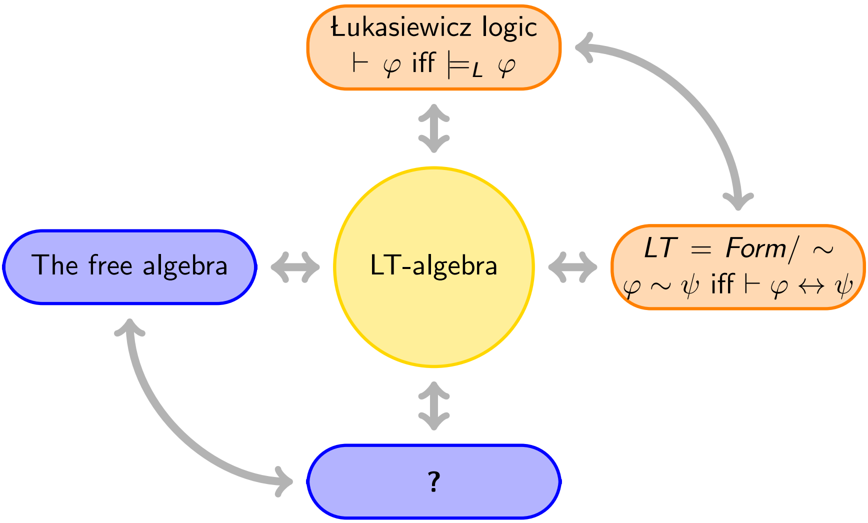
- $\mathcal{L}_2 = \mathcal{L}_\infty + ((\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi)$
- $\mathcal{L}_n = \mathcal{L}_\infty + Ax_n + \{Ax_k \mid k \in \{2, \ldots, (n-2)\}, k \nmid (n-1)\}$

Deduction rule: $\{\varphi, \varphi \rightarrow \psi\} \vdash \psi$

Logic and Algebra



Logic and Algebra



The Algebra of Logic

- Gr.C. Moisil, 1940: 3-valued and 4-valued Łukasiewicz algebras
- Gr.C. Moisil, 1941: n-valued *Łukasiewicz algebras*

Moisil's definition:
an element is uniquely characterized by
a sequence of Boolean nuances

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- C.C. Chang, 1958: *MV-algebras*,
  the algebraic structures corresponding to $\mathcal{L}_\infty$

Chang's definition
is inspired by the theory of lattice-ordered groups

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- R. Cignoli,1982: *proper n-valued Łukasiewicz algebras*, categorically equivalent to MV_n-algebras

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Moisil logic
and
Łukasiewicz-Moisil algebras

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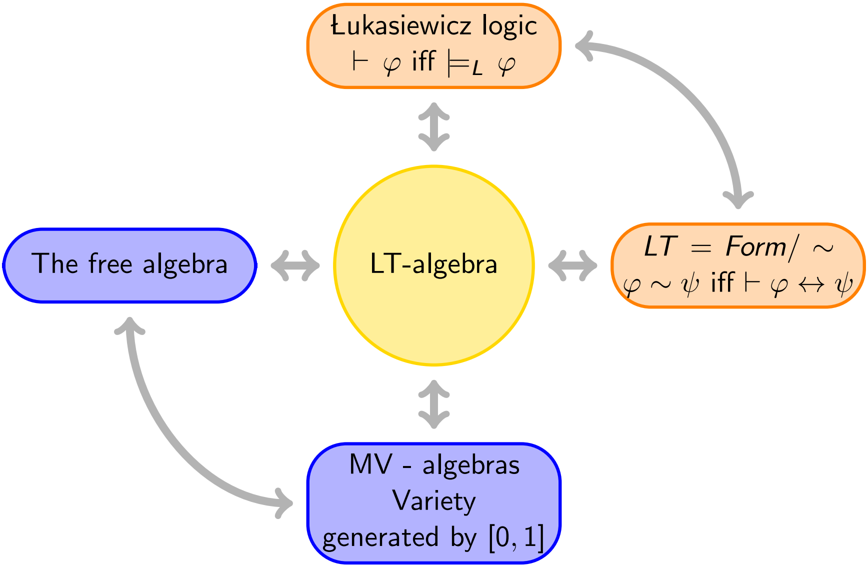

Lukasiewicz ∞-valued logic

$\mathcal{L}$

The connectives are $\{\rightarrow, \neg\}$

The axioms:

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$
2. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
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The deduction rule is *modus ponens*.

J. Łukasiewicz, A. Tarski, 1930

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Completeness. TFAE:

- φ is provable,
- $e(\varphi) = 1$ for any [0,1]-evaluation e.

A. Rose, J.B. Rosser, 1958

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$\mathcal{L} + (\neg\varphi \to \varphi) \to \varphi = CL$

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The algebra of Łukasiewicz logic

MV-algebra: $(A, \oplus, ^*, 0_A)$

1. $(A, \oplus, 0_A)$ abelian monoid,
2. $(x^*)^* = x$,
3. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
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for any x, $y \in A$.

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Derived operations:

$1_A = 0_A^*$,
$x \odot y = (x^* \oplus y^*)^*$,
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$(A, \vee, \wedge, 0_A, 1_A)$ is a bounded distributive lattice

$x \vee y = (y^* \oplus x)^* \oplus x$, $x \wedge y = (x^* \vee y^*)^*$ for any x, $y \in A$

The implication: $x \to y = 1_A$ iff $x \leq y$

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Boolean algebra=

MV-algebra s.t. $x \oplus x = x$

$(A, \vee, \wedge, 0_A, 1_A)$ is a bounded distributive lattice

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MV-algebras

Standard model: $([0,1], \oplus, ^*, 0)$

$x \oplus y = \min(x + y, 1)$,
$x^* = 1 - x$

Chang's completeness theorem: $\mathbb{MV}$=HSP([0,1])

C.C.Chang, 1959

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C.C.Chang, 1959

Chang's representation theorem

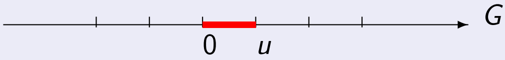
Any MV-algebra is a subdirect product of linearly ordered MV-algebras.

C.C.Chang, 1959

MV-algebras

$(G, +, 0, \leq)$ is a *lattice-ordered group* (*ℓ-group*) if
$(G, +, 0)$ group, $(G, \leq)$ lattice,
$x \leq y$ implies $x + z \leq y + z$ for any x, y, $z \in G$.

$u \in G$ is a **strong unit**: $u \geq 0$, for any $x \in G$ there is $n \geq 1$ s.t. $x \leq nu$.



$([0, u]_G, \oplus, ^*, 0)$ MV-algebra for any (G, u) ab. ℓu - group

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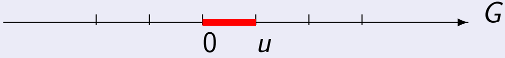

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Categorical equivalence

The category of MV-algebras is equivalent with the category of abelian ℓ-groups with strong unit with unit preserving homomorphism.
D. Mundici, 1986

Examples

- $[0,1]^X$, ${}^*[0,1]^X$ where X is a set

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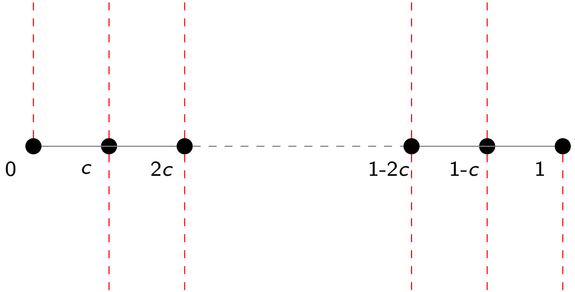
- $[(0,0),(1,0)]_{\mathbb{Z}\times_{lex}G}$ where G is an ℓ-group
  $[((0,0),0),((1,0),0)]_{(\mathbb{Z}\times_{lex}\mathbb{Z})\times_{lex}G}$ where G is an ℓ-group



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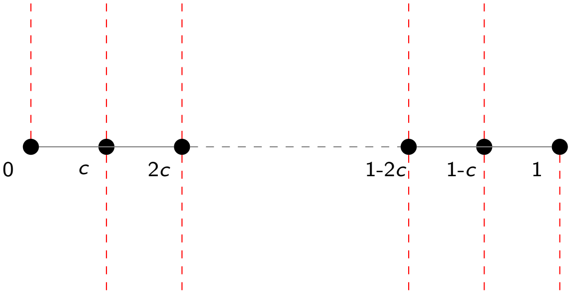

Non-semisimple MV-algebra = algebra with infinitesimals

Functional representation

- A MV-algebra, $\emptyset \neq I \subseteq A$ is an ideal if:
  $(x \in I,\ y \leq x \Rightarrow y \in I)$ and $(x,\ y \in I \Rightarrow x \oplus y \in I)$
- for any MV-algebra A, the maximal ideal space $MaxA$ with the spectral topology is a compact Hausdorff space
  (open sets: $r(I) = \{M \in MaxA \mid I \not\subseteq M\}$ for some ideal I).
- A is semisimple if $\bigcap\{M \mid M \in Max(A)\} = \emptyset$
- $C(MaxA) = \{f : MaxA \to [0,1] \mid f \text{ continuous}\}$

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L.P.Belluce, 1986

Any semisimple MV-algebra A is isomorphic with a separating MV-subalgebra of $C(MaxA)$ (with pointwise operations).

- $\iota \colon A \to C(MaxA)$ embedding
  $\forall\ M_1 \neq M_2\ \exists\ f \in \iota(A)\ (f(M_1) = 0 \text{ and } f(M_2) \neq 0)$

A. Di Nola, S.Sessa, 1995

- A σ-complete $\Rightarrow$ $MaxA$ basically disconnected
- A complete $\Rightarrow$ $MaxA$ extremally disconnected

Semantical and sintactical consequences in $\mathcal{L}$

For a set Θ of formulas, define
$\Theta^{\vdash}$ = sintactic consequences of Θ
$\Theta^{\models}$ = semantic consequences of Θ

Theorem

TFAE:

- $\Theta^{\vdash} = \Theta^{\models}$
- $\mathcal{L}(\Theta)$ (the Lindenbaum-Tarski algebra of Θ) is semisimple.

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R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, Algebraic foundations of many-valued reasoning, 2000.

P. Hájek, Metamathematics of fuzzy logic, 1998.

Di Nola's representation theorem, 1991

Theorem

Up to isomorphism, every MV-algebra A is an algebra of $[0,1]^*$-valued functions over some set, where $[0,1]^\star$ is an ultrapower of $[0,1]$, only depending on th cardinatlity of A.

- $[0,1]^\star$ is the unit interval of $\mathbb{R}^\star$ (non-standard reals)

Logic and games

- **Answerer**: chooses a number $x \in S$
- **Questioner**: asks Yes/No questions
- **Answerer**: Yes/No
- The **Answerer** is allowed to lie at most p times.

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Theorem

If α is a formula $\mathcal{L}_\infty$ t.f.a.e.:

- $\vdash_{\mathcal{L}_\infty} \alpha$,
- $e(\alpha) = 1$ for any S, $p \geq 0$ and $e : Form(\mathcal{L}) \to K(S, p)$ valuation, where $K(S, p)$ is an MV-algebra defined by Ulam game with a finite searching spaces S and p lies.

Mundici, 1991

MV-algebras are twofold structures

- generalization of Boolean algebras
- intervals in abelian lattice ordered groups with strong unit

The theory of MV-algebras is a possible answer to Birkhoff's problem: develop a common abstraction which includes Boolean algebras and lattice-ordered groups as special cases.

G. Birkhoff, Lattice Theory, 1973.

Fuzzy Sets

L.A. Zadeh, 1965

Clearly, the "class of all real numbers that are much greater then 1", or "the class of beautiful women", or "the class of tall men" do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such "imprecisely" defines classes play an important role in human thinking, particularly in the domains of pattern recognition, communication of information and abstraction.

. . .

The concept in question is that of a *fuzzy set*, that is, a "class" with a continuum of grades of membership.

A fuzzy subset of a set X is a function

$$\chi : X \to [0, 1].$$

Fuzzy Logic

L.A. Zadeh, 2004

- In many-valued logic, ML, truth is a matter of degree,
- In Fuzzy logic, FL:
  - everything is, or it is allowed to be, partial, i.e., a matter of degree,
  - everything is, or it is allowed to be, imprecise (approximate), linguistic, perception based.
- A source of confusion is that the label "fuzzy logic" is used in two different senses:
  - narrow sense: fuzzy logic is a logical system,
  - wide sense: fuzzy logic is coextensive with fuzzy set theory.

"FL in narrow sense"

P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, 1998

Fuzzy logic is the logic the continuous t-norms on $[0, 1]$.

A *t-norm* is an operation $* : [0, 1]^2 \to [0, 1]$ with the following properties:

- $*$ is commutative, associative, monotone
- $1 * x = x$ și $0 * x = 0$ for any $x \in [0, 1]$.

The most important continuous *t*-norms are:

$x * y = max(0, x + y - 1)$ (Łukasiewicz),
$x * y = min(x, y)$ (Gödel),
$x * y = x \cdot y$ (product)

Łukasiewicz logic with product

Łukasiewicz logic with product

The product in Łukasiewicz logic

$[0, 1]$ is closed to the real product.

- internal product: $([0, 1], \cdot, \oplus, ^*, 0)$

  A. Di Nola, A. Dvurečenskij, 2001, F. Montagna 2000: PMV-algebras

- external product: $([0, 1], \oplus, ^*, \{r | r \in [0, 1]\}, 0)$

  A. Di Nola, I. L., 2012: Riesz MV-algebras

- internal and external product: $([0, 1], \cdot, \oplus, ^*, \{r | r \in [0, 1]\}, 0)$

  S. Lapenta, I. L., 2015: fMV-algebras

Inspiration: lattice-ordered structures

$(G, +, 0, \leq)$ ℓ-group	$(G, +, 0)$ group, $(G, \leq)$ lattice, $x \leq y$ implies $x + z \leq y + z$
$(V, +, \{\mathbf{r} \mid r \in \mathbb{R}\}, 0, \leq)$ Riesz space	$(V, +, 0, \leq)$ abelian ℓ-group $(V, +, \{\mathbf{r} \mid r \in \mathbb{R}\}, 0)$ real vector space $x \leq y$ implies $r \cdot x \leq r \cdot y$ for $r \geq 0$
$(R, +, \cdot, 0, \leq)$ ℓ-ring	$(R, +, 0, \leq)$ abelian ℓ-group, $(R, +, \cdot, 0)$ ring $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for $z \geq 0$
$(A, +, \cdot, \{\mathbf{r} \mid r \in \mathbb{R}\}, 0, \leq)$ ℓ-algebra	$(A, +, \cdot, 0, \leq)$ ℓ-ring $(A, +, \{\mathbf{r} \mid r \in \mathbb{R}\}, 0, \leq)$ Riesz space $r(x \cdot y) = (rx) \cdot y = x \cdot (ry)$

f-ring (f-algebra) = subdirect product of chains

MV-algebras are intervals in ℓu-groups

MV-algebra: $(A, \oplus, ^*, 0_A)$

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for any $x, y \in A$.

$([0, u]_G, \oplus, ^*, 0)$ MV-algebra for any (G, u) ab. ℓu - group

$x \oplus y = (x + y) \wedge u$, $x^* = u - x$ for any $x, y \in G$.

What linearity means in the theory of MV-algebras?

Intuition:

If $A = [0, u]_G$ then $x \oplus_A y = x +_G y$ iff $x \odot y = 0$ iff $x \leq y^*$.

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Intuition:

If $A = [0, u]_G$ then $x \oplus_A y = x +_G y$ iff $x \odot y = 0$ iff $x \leq y^*$.

$\omega : A \to B$ function, A and B are MV-algebras

TFAE:

- $x \odot y = 0$ implies $\omega(x) \odot \omega(y) = 0$ and $\omega(x \oplus y) = \omega(x) \oplus \omega(y)$,
- the following properties hold for any $x, y \in A$:
  (l1) $x \leq y$ implies $\omega(x) \leq \omega(y)$,
  (l2) $\omega(x \odot (x \wedge y)^*) = \omega(x) \odot \omega(x \wedge y)^*$.

Definition

ω is linear if it satisfies the above properties.

$\beta : A \times B \to C$ is bilinear if it is linear in each argument.

Overview on Łukasiewicz logic with product

Algebra	*ℓ-structure*	
MV-algebras	ℓ-groups	$HSP([0,1])$
Riesz MV-algebras	Riesz spaces	$HSP([0,1])$
Product MV-algebras	f-rings	$HSP([0,1])$
fMV-algebras	f-algebras	$HSP([0,1])$

$HSP([0,1])$ is a proper subvariety

Riesz MV-algebras

Riesz MV-algebra $(A, \oplus, ^*, \{r \mid r \in [0,1]\}, 0_A)$

1. $(A, \oplus, ^*, 0_A)$ is an MV-algebra,
2. $r(x \odot y^*) = (rx) \odot (ry)^*$,
3. $(r \odot q^*) \cdot x = (rx) \odot (qx)^*$,
4. $r(qx) = (rq)x$,
5. $1x = x$,

r, $q \in [0,1]$ and x, $y \in A$, where $x \odot y = (x^* \oplus y^*)^*$.

in: A. Di Nola, I.L., 2011, 2014, A. Di Nola, S.Lapenta, I.L., 2018

fMV-algebras

fMV-algebra $(A, \oplus, \cdot, ^*, \{r \mid r \in [0,1]\}, 0_A)$

1. $(A, \oplus, ^*, \{r \mid r \in [0,1]\}, 0_A)$ is a Riesz MV-algebra
2. $z \cdot (x \odot (x \wedge y)^*) = (z \cdot x) \odot (z \cdot (x \wedge y))^*$
3. $(x \odot (x \wedge y)^*) \cdot z = (x \cdot z) \odot ((x \wedge y) \cdot z)^*$
4. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
5. $(z \cdot (x \odot y^*)) \wedge (y \odot x^*) = 0_A$
6. $((x \odot y^*) \cdot z) \wedge (y \odot x^*) = 0_A$
7. $r(x \cdot y) = (rx) \cdot y = x \cdot (ry)$

$r \in [0,1]$, and $x, y, z \in A$ where $x \odot y = (x^* \oplus y^*)^*$.

in: S.Lapenta, I.L., 2016

A unifying framework

$$
\begin{tikzcd}
\mathbf{uAG} \arrow[dr, "\Gamma"] & & & \mathbf{uRS} \arrow[lll, "U"'] \arrow[dl, "\Gamma_{\mathbb{R}}"'] \\
& \mathbf{MV} & \mathbf{RMV} \arrow[l, "U"'] & \\
& \mathbf{PMV} \arrow[u, "U"'] & \mathbf{fMV} \arrow[l, "U"'] \arrow[u, "U"] & \\
\mathbf{uR} \arrow[uuu, "U"'] \arrow[ur, "\Gamma_{(\cdot)}"'] & & & \mathbf{ufAlg} \arrow[lll, "U"'] \arrow[ul, "\Gamma_{(\cdot),\mathbb{R}}"] \arrow[uuu, "U"]
\end{tikzcd}
$$

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\end{tikzcd}
$$

PROBLEM: find adjunctions and close the diagrams

A unifying framework

$$
\begin{tikzcd}
\mathbf{uAG_a} \arrow[dr, "\Gamma"] & & & \mathbf{uRS_a} \arrow[lll, "U"'] \arrow[dl, "\Gamma_{\mathbb{R}}"'] \\
& \mathbf{MV_{ss}} & \mathbf{RMV_{ss}} \arrow[l, "U"] & \\
& \mathbf{PMV_{ss}} \arrow[u, "U"'] & \mathbf{fMV_{ss}} \arrow[l, "U"'] \arrow[u, "U"] & \\
\mathbf{uR_a} \arrow[uuu, "U"'] \arrow[ur, "\Gamma_{(\cdot)}"'] & & & \mathbf{ufAlg_a} \arrow[lll, "U"'] \arrow[ul, "\Gamma_{(\cdot),\mathbb{R}}"] \arrow[uuu, "U"]
\end{tikzcd}
$$

PROBLEM: find adjunctions and close the diagrams
SOLUTION: for semisimple stuctures,
using the MV-algebraic tensor product

One diagram ... "to rule them all"

$$\begin{tikzcd}
\mathbf{uAG_a} \arrow[rrr, bend left, "(\mathbb{R},1)\otimes_a -"] \arrow[dr, "\Gamma"] \arrow[ddd, bend right, "\mathbf{T}_l"'] & & & \mathbf{uRS_a} \arrow[lll] \arrow[dl, "\Gamma_{\mathbb{R}}"] \arrow[ddd, bend left, "\mathbf{T}_l"] \\
& \mathbf{MV_{ss}} \arrow[r, bend left, "{[0,1]\otimes_s -}"] \arrow[d, bend right, "\mathbf{T}"'] & \mathbf{RMV_{ss}} \arrow[l] \arrow[d, bend left, "\mathbf{T}"] & \\
& \mathbf{uPMV_{ss}} \arrow[u] \arrow[r, bend right, "{[0,1]\otimes_s -}"'] & \mathbf{ufMV_{ss}} \arrow[l] \arrow[u] & \\
\mathbf{uR_a} \arrow[uuu] \arrow[ur, "\Gamma_{(\cdot)}"] \arrow[rrr, bend right, "(\mathbb{R},1)\otimes_a -"'] & & & \mathbf{ufMV_a} \arrow[lll] \arrow[uuu] \arrow[ul, "\Gamma_{(\cdot),\mathbb{R}}"']
\end{tikzcd}$$

S. Lapenta, I.L., 2015, 2016