Special Topics in Many-Valued Logics

Part II - Riesz MV-algebras and their logic

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Special Topics in Computer Science

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The product in Łukasiewicz logic

- internal product: $([0,1], \oplus, \cdot, ^*, 0)$

  A. Di Nola, A. Dvurečenskij, 2001: PMV-algebras

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- scalar multiplication: $([0,1], \oplus, ^*, \{r | r \in [0,1]\}, 0)$

  A. Di Nola, I.L., 2011: Riesz MV-algebras

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- internal product and scalar multiplication:
  $([0,1], \oplus, \cdot, ^*, \{r | r \in [0,1]\}, 0)$

  S. Lapenta, I.L., 2014: fMV-algebras

MV-algebras with product

Category	*MV-algebras with product*	*unital ℓ-structures*	*Standard model*
MV	MV-algebras	ℓ-groups	$HSP([0,1])$
RMV	Riesz MV-algebras	Riesz spaces	$HSP([0,1])$
PMV	Product MV-algebras	f-rings	$PMV^+ = ISP([0,1])$
fMV	fMV-algebras	f-algebras	$fMV^+ = ISP([0,1])$

MV-algebras with product

Logic	*MV-algebras with product*	*unital ℓ-structures*	*Standard model*
$\mathcal{L}_\infty$	MV-algebras	ℓ-groups	HSP$([0,1])$
$\mathbb{R}\mathcal{L}$	Riesz MV-algebras	Riesz spaces	HSP$([0,1])$
$\mathcal{PL}$	Product MV-algebras	f-rings	PMV$^+$ =ISP$([0,1])$
$f\mathcal{L}$	fMV-algebras	f-algebras	fMV$^+$ $=$ ISP$([0,1])$

A. Di Nola, I.L., 2011, 2014

Riesz MV-algebra $(A, \oplus, ^*, \{r \mid r \in [0,1]\}, 0_A)$

1. $(R, \oplus, ^*, 0_A)$ is an MV-algebra,
2. $r(x \odot y^*) = (rx) \odot (ry)^*$,
3. $(r \odot q^*) \cdot x = (rx) \odot (qx)^*$,
4. $r(qx) = (rq)x$,
5. $1x = x$,

r, $q \in [0,1]$ and x, $y \in A$, where $x \odot y = (x^* \oplus y^*)^*$.

Riesz MV-algebras and Riesz spaces

- For any Riesz MV-algebra A there is al Riesz space with strong unit (V, u) such that $A \simeq [0,u]_V$.
- The category of Riesz MV-algebras is equivalent with the category of Riesz spaces with strong unit.

The logic $\mathbb{R}\mathcal{L}$

The connectives are $\{\rightarrow, \neg\} \cup \{\sigma_r \mid r \in [0,1]\}$
The axioms:

1. the axioms of $\mathcal{L}$
2. $\sigma_r(\varphi \rightarrow \psi) \leftrightarrow (\sigma_r\varphi \rightarrow \sigma_r\psi)$
3. $\sigma_{(r\odot q^*)}\varphi \leftrightarrow (\sigma_q\varphi \rightarrow \sigma_r\varphi)$
4. $\sigma_r\sigma_q\varphi \leftrightarrow \sigma_{(rq)}\varphi$
5. $\sigma_1\varphi \leftrightarrow \varphi$,

where φ, ψ, χ are formulas and r, $q \in [0,1]$.
The deduction rule is *modus ponens*.

Theorem

The Lindenbaum-Tarski algebra is a Riesz MV-algebra.

The logic $\mathbb{R}\mathcal{L}$

Completeness. TFAE:

- φ is provable,
- $e(\varphi) = 1$ for any $[0,1]$-evaluation.

Theorem

RMV=HSP([0,1])

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For $r \in [0,1]$ we set $\mathbf{r} := \neg(\sigma_r(\neg\theta))$ where $\vdash \theta$

Pavelka completeness

If φ is a formula of $\mathbb{R}$L , we define:

- the *truth degree* of φ, by
  $\| \varphi \| = \min\{e(\varphi) \mid e \text{ is an evaluation}\}$,
- the *provability degree* of φ, by
  $| \varphi | = \max\{r \in [0,1] \mid \vdash \mathbf{r} \to \varphi\}$,

then $| \varphi | = \| \varphi \|$.

Normal form theorems

- φ formula with n variables $\mapsto f_\varphi : [0,1]^n \to [0,1]$

A function $f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(\mathbb{Z})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

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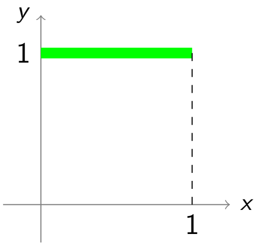
Free MV-algebra $MV_n \simeq Lind_{\mathcal{L},n}$ [R. McNaughton, 1951]

$Free_{MV}(n) = \{f_\varphi : [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathcal{L}\} = \mathrm{PWL}_u(\mathbb{Z})$

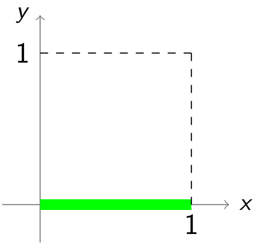
Free Riesz MV-algebra $RMV_n \simeq Lind_{\mathbb{R}\mathcal{L},n}$

$Free_{RMV}(n) = \{f_\varphi : [0,1]^n \to [0,1] \mid \varphi \text{ formula of } \mathbb{R}\mathcal{L}\} = \mathrm{PWL}_u(\mathbb{R})$

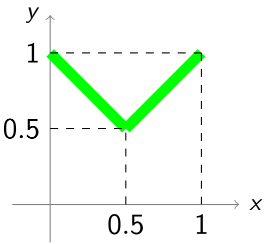
Elements of $Free_{MV}(1)$



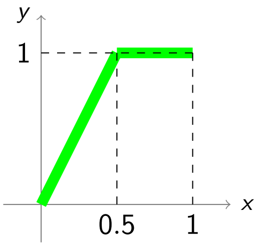
$f(x) = x \oplus x^*$



$f(x) = x \odot x^*$



$f(x) = x \vee x^*$



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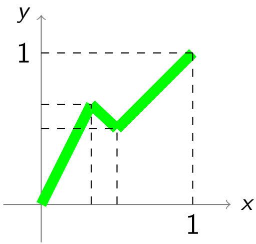
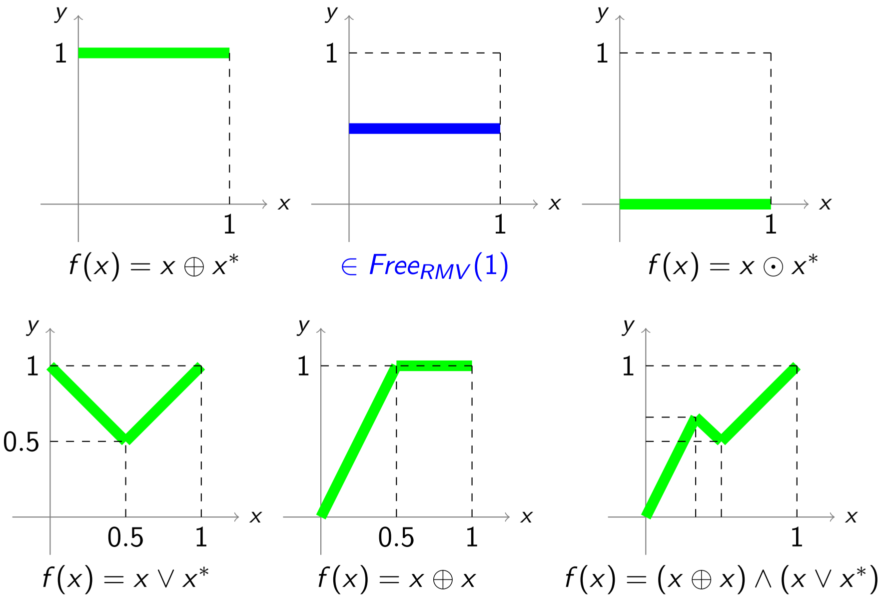


$f(x) = (x \oplus x) \wedge (x \vee x^*)$

Elements of $Free_{MV}(1)$ and $Free_{RMV}(1)$



Overview

- Riesz MV-algebras are a variety,
- Riesz MV-algebras have a simple axiomatization,
- the logic $\mathbb{R}\mathcal{L}$ is "classically" developed,
- the logic $\mathbb{R}\mathcal{L}$ has standard completeness and normal-form theorem,
- Riesz MV-algebras categorically equivalent with Riesz spaces with strong unit.

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... most of the standard real function spaces are vector lattices, and in a very natural way.

G. Birkhoff, Lattice Theory, 1973

Banach lattices: M-spaces, L-space

A Banach lattice $(R, \|\cdot\|)$ is an

- M-space: $\|x \vee y\| = \|x\| \vee \|y\|$ whenever $x,\ y \geq 0$
- L-space: $\|x + y\| = \|x\| + \|y\|$ whenever $x,\ y \geq 0$

Examples

- $C(X, \mathbb{R})$ is an M-space for any compact Hausdorff space X.
- $L_1(\mu)$ is an L-space for any measure space (X, Ω, μ).

Kakutani's theorems

- For any M-space with strong unit (M, u) there exists a compact Hausdorff space X such that (M, u) is isometrically Riesz isomorphic with $C(X, \mathbb{R})$.
- For any L-space with strong unit (L, u) there exists a measure space (X, Ω, μ) such that L is isometrically Riesz isomorphic with $L^1(\mu)$.

On any semisimple Riesz MV-algebra A define

$$\|x\|_u = \inf\{r \in [0,1] \mid x \leq r \cdot 1\} \text{ for any } x \in A \text{ (unit norm)}$$

Note that $\|x \vee y\|_u = \|x\|_u \vee \|y\|_u$ for any $x,y \in A$.

Unit intervals in M-spaces

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Norm-complete Riesz MV-algebras and M-spaces

Any Riesz MV-algebra which is norm-complete w.r.t $\|\cdot\|_u$ is isomorphic with the unit interval of an M-space with strong unit.

The category of Riesz MV-algebras which are norm-complete w.r.t $\|\cdot\|_u$ is equivalent with the category of unital M-spaces with Riesz homomorphisms.

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Norm-complete Riesz MV-algebras and compact Hausdorff spaces

A Riesz MV-algebra A is norm-complete w.r.t. $\|\cdot\|_u$ iff $A \simeq C(Max(A),[0,1])$.

The category of Riesz MV-algebra which are norm-complete w.r.t $\|\cdot\|_u$ with Riesz MV-algebra homomorphisms is:

- dually equivalent with the category of compact Hausdorff spaces with continuous functions,
- equivalent with the category of C^*-algebras.

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I.L.: similar results for L-spaces

Logic and analysis

$\mathbb{R}\mathcal{L}$ the logic of Riesz MV-algebras
$[\varphi]$ in $Lind_{\mathbb{R}\mathcal{L},n}$

$\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0,1]^n\}$
$(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is a normed space

Norm-completion of the Lindenbaum-Tarski algebra

The norm-completion of the normed space $(Lind_{\mathbb{R}\mathcal{L},n}, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$.

Approximation of continuous functions

For any continuous function $f : [0,1]^n \to [0,1]$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathbb{R}\mathcal{L}$ such that $\lim_n f_{\varphi_n} = f$.

For $r \in [0,1]$ we set $\mathbf{r} := \neg(\sigma_r(\neg\theta))$ where $\vdash \theta$

Uniform limit (inspired by [X. Caicedo, LATD'08])

A formula φ is *the uniform limit* of the sequence $(\varphi_n)_n$ in $\mathbb{R}\mathcal{L}$ if for any $r < 1$ there is k such that for any $n \geq k$: $\vdash \mathbf{r} \to (\varphi \leftrightarrow \varphi_n)$.
We write $\lim_n \varphi_n = \varphi$.

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Theorem. TFAE:

- $\lim_n \varphi_n = \varphi$,
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- there exists an increasing sequence $(r_n)_n$ in $[0,1]$ such that $\bigvee_n r_n = 1$ and $\vdash \eta_{r_n} \to (\varphi \leftrightarrow \varphi_n)$ for any n

The logic $\mathbb{Q}\mathcal{L}$

$\mathbb{Q}\mathcal{L}$

The language is $\{\to, \neg\} \cup \{\sigma_r \mid r \in [0,1] \cap \mathbb{Q}\}$
The axioms are similar with the ones of $\mathbb{R}\mathcal{L}$.
Completeness w.r.t. $[0,1] \cap \mathbb{Q}$.

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Rational Łukasiewicz logic [B. Gerla, 2001]

The language is $\{\rightarrow, \neg\} \cup \{\delta_n \mid n \in \mathbb{N}\}$
I. the axioms of Ł:
II. the following formulas are axioms for any $n \geq 2$:
$\varphi \leftrightarrow n\delta_n\varphi$
$(\varphi \rightarrow n\psi) \rightarrow (\delta_n\varphi \rightarrow \psi)$

$\mathbb{Q}\mathcal{L}$ and *Rational Łukasiewicz logic* are equivalent.

The logic $\mathbb{Q}\mathcal{L}$

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The axioms are similar with the ones of $\mathbb{R}\mathcal{L}$.

DMV-algebras

The models of $\mathbb{Q}\mathcal{L}$ are defined as Riesz MV-algebras, but we have only rational scalars.
They are categorically equivalent with:

- divisible MV-algebras,
- $\mathbb{Q}$-vector lattices with strong unit

$\mathbf{DMV} = HSP([0,1]_{\mathbb{Q}})$

Formulas as limits

For any formula φ of $\mathbb{R}\mathcal{L}$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathbb{Q}\mathcal{L}$ such that $\lim_n \varphi_n = \varphi$.

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$\mathbb{R}\mathcal{L}^*$ is the logic obtained from $\mathbb{R}\mathcal{L}$ by adding deduction rules

$$(\star) \qquad \text{if } \varphi = \lim_m \varphi_m \quad \text{then} \quad \frac{\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots}{\varphi}$$

Limits and deduction

Let φ be a formula of $\mathbb{R}\mathcal{L}$. There exists a sequence of formulas $\Theta = \{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathit{Form}_{\mathbb{Q}\mathcal{L}}$ such that:

- $\lim_n \varphi_n = \varphi$,
- $\mathit{Thm}(\varphi, \mathbb{R}\mathcal{L}^*) = \mathit{Thm}(\Theta, \mathbb{R}\mathcal{L}^*)$.

Theorem. TFAE:

- $\lim_n \varphi_n = \varphi$,
- $\lim_n f_{\varphi_n} = f_\varphi$ (uniform convergence),
- there exists an increasing sequence $(r_n)_n$ in $[0,1]$ such that $\bigvee_n r_n = 1$ and $\vdash \eta_{r_n} \to (\varphi \leftrightarrow \varphi_n)$ for any n

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How to axiomatize $C([0,1]^n)$?

Marra V. and Reggio L., 2016

- They introduce δ-algebra: MV-algebra endowed with an infinitary operator δ such that, for sequences in $C(X)$, is $\sum_n \frac{f_n}{2^n}$.
- The category of compact Hausdorff spaces is dual to the category of δ-algebras with δ-preserving MV-algebra morphisms.

- Hence, δ-algebras are equivalent to norm-complete Riesz MV-algebras,

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- Hence, δ-algebras are equivalent to norm-complete Riesz MV-algebras,
- In $\mathbf{R}\mathcal{L}$: given $\{\varphi_n\}_n$ we define the sequence $\sigma_1 = \Delta_{\frac{1}{2}}\varphi_1$, $\sigma_2 = \Delta_{\frac{1}{2}}\varphi_1 \oplus \Delta_{\frac{1}{2^2}}\varphi_2, \ldots$ and we set $\delta(\varphi_1, \varphi_2, \cdots) = lim_n \sigma_n$.

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What happens if we close to $\bigvee_n$ and $\bigwedge_n$?

- Language: $\{\to, \neg\} \cup \{\nabla_r\}_{r\in[0,1]} \cup \bigvee$

- Axioms: the ones of $\mathbb{R}\mathcal{L}$ and

  (S1) $\varphi_k \to \bigvee_{n\in\mathbb{N}} \varphi_n$, for any $k \in \mathbb{N}$

- Deduction rules: Modus Ponens and

(SUP) $$\frac{(\varphi_1 \to \psi), \ldots, (\varphi_k \to \psi) \ldots}{\bigvee_{n\in\mathbb{N}} \varphi_n \to \psi}$$

The infinitary logic $\mathcal{IRL}$

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Following Karp C. R., *Languages with expressions of infinite length*, North-Holland Pub. Co., 1964

The infinitary logic $\mathcal{IRL}$

- Models are σ-complete Riesz MV-algebras
- $\mathcal{IRL}$ is complete wrt to all models
- $Lind_{\mathcal{IRL}}$ is the Dedekind σ-completion of $Lind_{\mathbb{R}\mathcal{L}}$

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A logic for quasi-Stonean spaces

Since any σ-complete Riesz MV-algebra is norm-complete (w.r.t. unit norm), any model of $\mathcal{IRL}$ has the form $C(X)$ where X is quasi-Stonean (basically disconnected compact Hausdorff space). In particular, $Lind_{\mathcal{IRL}}$ has this form.

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Riesz tribe

A *Riesz tribe* over X is a Riesz MV-algebra of $[0,1]$-valued functions over X that are closed under pointwise countable suprema.

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The Loomis-Sikorski Theorem

Let $A \subseteq C(X)$ be a σ-complete Riesz MV-algebra, where $X = Max(A)$. Then there exist a Riesz tribe $\mathcal{T}$ and a σ-homomorphism of $\mathcal{T}$ onto A.

Representation using tribes

$Lind_{\mathcal{IRL},n}$ is isomorphic with the Riesz tribe on $[0, 1]^n$ generated by the projection functions.

Słomiński J., *The theory of abstract algebras with infinitary operations*, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa (1959)

Infinitary variety

The class of Dedekind σ-complete Riesz MV-algebras is the infinitary variety $HSP([0,1])$.

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Standard completeness for $\mathcal{IRL}$

If φ is a formula of $\mathcal{IRL}$, then $\vdash_{IRL} \varphi$ if and only if $e(\varphi) = 1$ for any evaluation $e : FORM_{IRL} \to [0,1]$.

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- the logic $\mathbb{R}\mathcal{L}$ has standard completeness and normal-form theorem,
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Riesz MV-algebras and their logic

- Riesz MV-algebras categorically equivalent with Riesz spaces with strong unit,
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- Neural networks as formulas of $\mathbb{R}\mathcal{L}$.

Multilayer perceptron

A multilayer perceptron with l hidden layers, n inputs and one output can be represented as a function $F : [0,1]^n \to [0,1]$ such that

$$F(x_1, \dots, x_n) = \phi\left(\sum_{k=1}^{n^{(l)}} \omega_{ok}^{l} \phi\left(\sum_{j=1}^{n^{(l-1)}} \omega_{kj}^{l-1} \phi\Big(\dots \left(\sum_{i=1}^{n} \omega_{li}^{1} x_i + b_i\right) \dots \Big)\right)\right),$$

where

- $\phi : \mathbb{R} \to [0,1]$ is the activation function (monotone-nondecreasing continuous function)
- ω_{ok}^{l} is the synaptic weight from neuron k in the l-th hidden layer to the single output neuron o,
- ω_{kj}^{l-1} is the synaptic weight from neuron j in the $(l-1)$-th hidden layer to neuron k in the l-th hidden layer,

  and so on for the other synaptic weights.

S. Haykin, Neural Neworks - A Comprehensive Foundation, 1999.

Neural networks as formulas of $\mathbb{R}\mathcal{L}$

Let $\mathcal{N}_n$ the class of multilayer perceptrons where the activation function is the piecewise linear function $\varrho(x) = \max(\min(1, x), 0)$, and the synaptic weights are real numbers.

Theorem: $\mathcal{N}_n = Free_{RMV}(n)$.

(a) For every $l, n, n^{(2)}, \ldots, n^{(l)} \in \mathbb{N}$, and $\omega_{ij}, b_i \in \mathbb{R}$, the function $F : [0,1]^n \longmapsto [0,1]$ defined as

$$F(x_1, \ldots, x_n) = \varrho\left(\sum_{k=1}^{n^{(l)}} \omega_{ok}\varrho\left(\sum_{j=1}^{n^{(l-1)}} \omega_{kj}\varrho\Big(\ldots\ (\sum_{i=1}^{n} \omega_{li}x_i + b_i)\ldots\Big)\right)\right),$$

belongs to $Free_{RMV}(n)$.

(b) For any $f \in Free_{RMV}(n)$, there exist $l, n, n^{(2)}, \ldots, n^{(l)} \in \mathbb{N}$, and $\omega_{ij}, b_i \in \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = \varrho\left(\sum_{k=1}^{n^{(l)}} \omega_{ok}\varrho\left(\sum_{j=1}^{n^{(l-1)}} \omega_{kj}\varrho\Big(\ldots\ (\sum_{i=1}^{n} \omega_{li}x_i + b_i)\ldots\Big)\right)\right).$$

Di Nola A., B. Gerla, I.L., 2013

Neural networks as formulas of $\mathbb{R}\mathcal{L}$

Problem

For $f : [0,1]^n \to \mathbb{R}$ an affine function, find a formula φ of $\mathbb{R}\mathcal{L}$ such that $f_\varphi = \varrho \circ f$.

```
function Formula((r_1, i_1), ..., (r_m, i_m))
{
(F1)  if r_k ≤ 0 for any k ∈ {1, ..., m} then return(⊥);
(F2)  find k ∈ {1, ..., m} such that r_k > 0;
      if i_k = 0 then ψ := r_k⊤ else ψ := r_k x_{i_k};
(F3)  if m = 1 then return(ψ);
(F4)  φ =Formula((r_1, i_1), ..., (r_{k-1}, i_{k-1}), (r_{k+1}, i_{k+1}), ..., (r_m, i_m)) ;
      χ =Formula((-r_1, i_1), ..., (-r_{k-1}, i_{k-1}), (-r_{k+1}, i_{k+1}), ..., (-r_m, i_m))
      return((φ ⊕ ψ) ⊙ ¬χ)
}
```

Di Nola A., B. Gerla, I.L., 2013

We illustrate how the algorithm works on a simple example.

For $f : [0,1]^2 \to [0,1]$, $f(x_1, x_2) = x_2 - 0.3x_1$ we call the function

```
function Formula((1,2),(−0.3,1))
{
(F2) k = 1, r_k = 1, i_k = 2; ψ := 1x_2;
(F4) φ =Formula((−0.3,1)) ; χ =Formula((0.3,1)) ;
return((φ ⊕ ψ) ⊙ ¬χ)
}
```

Denote $\perp = \neg(\varphi \to \varphi)$. One can easily see that $\varphi = \perp$ and $\chi = \mathbf{0.3}x_1$, so the function returns

$$(\varphi \oplus \psi) \odot \neg\chi = (\perp \oplus \mathbf{1}x_2) \odot \neg\mathbf{0.3}x_1$$

which is logically equivalent with $x_2 \odot \neg\mathbf{0.3}x_1$.

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