Special Topics in Many-Valued Logics

Part III - Many-Valued Logic and Games

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Special Topics in Computer Science

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Ulam game with lies

- **Answerer**: chooses a number $x \in S$
- **Questioner**: asks Yes/No questions
- **Answerer**: Yes/No
- The **Answerer** is allowed to lie at most p times.

Theorem

If α is a formula $\mathcal{L}_\infty$ t.f.a.e.:

- $\vdash_{\mathcal{L}_\infty} \alpha$,
- $e(\alpha) = 1$ for any S, $p \geq 0$ and $e : Form(\mathcal{L}) \to K(S, p)$ valuation, where $K(S, p)$ is an MV-algebra defined by Ulam game with a finite searching spaces S and p lies.

Mundici, 1991

Ulam game with lies

- S finite (searching space), p number of lies
- (question,answer): Q_X^a with $X \subseteq S$, $a \in L_2$
- The knowledge after Q_X^a is $K_X^a : S \to L_{p+2}$,
  $K_X^a(x) = 1$ if ($a = 1$ and $x \in X$) or ($a = 0$ and $x \notin X$)
  $K_X^a(x) = \frac{p}{p+1}$, otherwise

  (*we do not reject a possible solution*)
- A game is $\sigma := Q_{X_1}^{a_1} \cdots Q_{X_n}^{a_n} \in Ulam(S, p)$
- The state of knowledge $K_\sigma : S \to L_{p+2}$, $K_{\sigma_1\sigma_2} = K_{\sigma_1} \odot K_{\sigma_2}$
- $K(S, p) = \{K_\sigma \mid \sigma$ game in $Ulam(S, p)\}$ is an MV-algebra

Mundici, 1991

Theorem. If α is a formula $\mathcal{L}_{p+2}$ the following are equivalent:

- $\vdash_{\mathcal{L}_{p+2}} \alpha$,
- $e(\alpha) = 1$ for any S and $e : Form(\mathcal{L}) \to K(S, p)$ valuation.

D. Mundici, 1991

$$\frac{\textit{Game}}{\textit{Logic}} = \frac{\text{Ulam game with } p \text{ lies}}{(p+2)\text{-valued Łukasiewicz logic}}$$

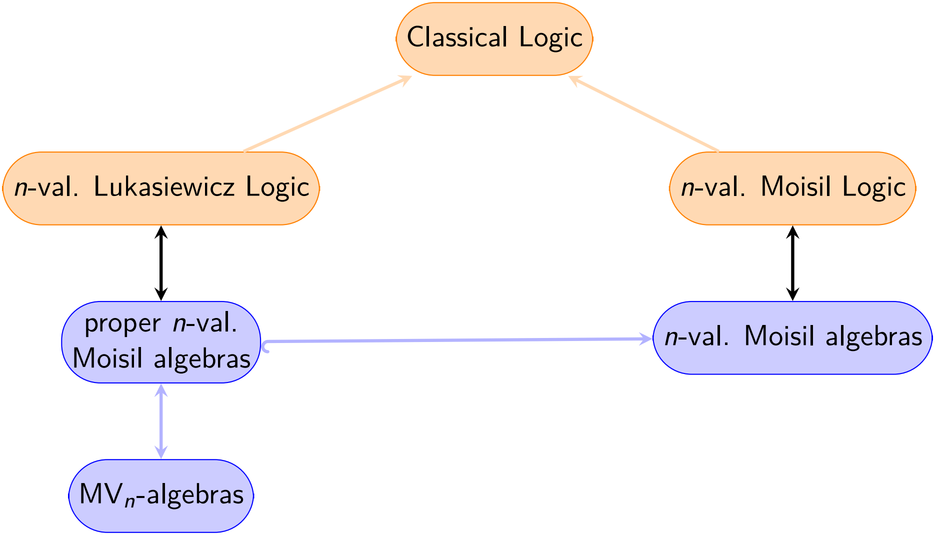
Logic and games

Game theory is strictly related to logic. For example, proving a formula can be regarded as a game between two players, the prover, who tries to prove the formula, and the opponent, who tries to attack the prover's attempts. However, in this case the games are ad hoc, in the sense that they are not interesting in themselves, their interest is based on the fact that they constitute a good semantics for proofs. Rényi-Ulam game with lies is an exception: it is an interesting game with several applications, and at the same time it constitutes a very natural semantics for Łukasiewicz logic. This suggests the following problem: for any of the games listed above, try to find a logic of which the game is a semantics. In other words, for G being a game, try to solve the equation:

$$\frac{\text{Ulam game with lies}}{\text{Łukasiewicz logic}} = \frac{G}{x}$$

C. Marini, F. Montagna, 2005.

Many-valued logics



$(n+1)$-valued Moisil algebras

$(L, \vee, \wedge, ^*, \varphi_1, \ldots, \varphi_n, 1)$

1. $(L, \vee, \wedge, ^*)$ De Morgan algebra
2. $\varphi_i(\varphi_j(x)) = \varphi_j(x)$
3. $\varphi_i(x \vee y) = \varphi_i(x) \vee \varphi_i(y)$
4. $\varphi_i(x) \vee (\varphi_i(x))^* = 1$
5. $\varphi_i(x^*) = (\varphi_{n+1-i}(x))^*$
6. $\varphi_1(x) \leq \cdots \leq \varphi_n(x)$
7. The determination principle
   $\varphi_i(x) = \varphi_i(y)$ for any $1 \leq i \leq n \Rightarrow x = y$

Heyting implication

$x \Rightarrow y := y \vee \bigwedge_{i=1}^{n}(\varphi_i(x)^* \vee \varphi_i(y))$

V. Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu, 1991

Standard model

L_{n+1}

	0	$\frac{1}{n}$	$\frac{2}{n}$	$\dots$	$\frac{n-1}{n}$	1
φ_1	0	0	0	$\dots$	0	1
φ_2	0	0	0	$\dots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
φ_{n-1}	0	0	1	$\dots$	1	1
φ_n	0	1	1	$\dots$	1	1

$$\mathbf{Moisil_{n+1}} = \mathrm{HSP}(L_{n+1})$$

$(n+1)$-valued Moisil algebra L

The Boolean center $B(L)$

$B(L) = \{x \in L \mid \varphi_i(x) = x \text{ for any } i\}$ is a Boolean algebra.

Determination principle

$$L \ni x \mapsto \varphi_1(x) \leq \cdots \leq \cdots \varphi_n(x) \in B(L)$$

$$x = y \text{ iff } \varphi_i(x) = \varphi_i(y) \text{ for any } i \in \overline{1,n}$$

Any element is characterized by n Boolean nuances.

Generalization from elements to algebras

$$L \mapsto (B(L), I_1, \ldots, I_{n-1})$$

$$I_1, \ldots, I_{n-1} \subseteq B(L) \text{ are Boolean ideals}$$

I.L., 2008; D. Diaconescu, I.L. 2015

$(n+1)$-valued Moisil logic $\mathcal{M}_{n+1}$

Connectives: $\vee, \neg, \Rightarrow, \varphi_1, \ldots, \varphi_n$

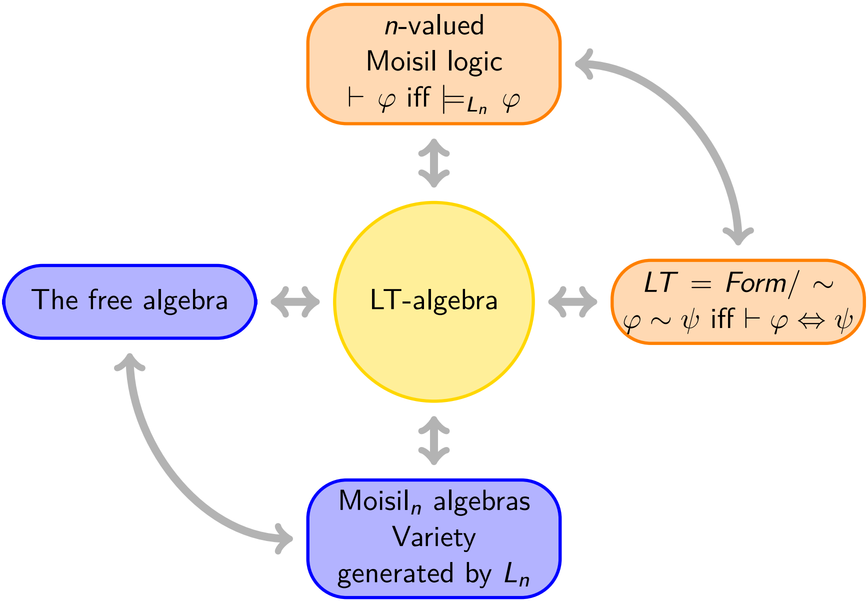
Axioms: (A1) ... (A18)

Deduction rules: $\{\alpha, \alpha \to \beta\} \vdash \beta$ and $\{\alpha\} \vdash \varphi_1\alpha$

The Lindenbaum-Tarski algebra is an $(n+1)$-valued Moisil algebra.

Completeness. The following are equivalent:

- $\vdash \alpha$
- $e(\alpha) = 1$ for any L_{n+1}-evaluation e.


n-valued
Moisil logic
$\vdash \varphi$ iff $\models_{L_n} \varphi$
The free algebra
LT-algebra
$LT = Form/\sim$
$\varphi \sim \psi$ iff $\vdash \varphi \Leftrightarrow \psi$
Moisil$_n$ algebras
Variety
generated by L_n

$(n+1)$-valued Moisil algebras

Example: $B^{[n]}$, where B is a Boolean algebra
$B^{[n]} = \{(b_1, \ldots, b_n) \mid b_1 \geq \cdots \geq b_n\}$
$\varphi_k(b_1, \ldots, b_n) = (b_k, \ldots, b_k)$

Note that $L_2^{[n]} \simeq L_{n+1}$

Embedding Theorem

The map $\Phi : L \to B(L)^{[n]}$, $\Phi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ is an embedding.
Moreover, Φ is an isomorphism iff L is an $n+1$-valued Post algebra

Consequence

The functor $B : \mathbf{Moisil_{n+1}} \to \mathbf{Bool}$ has a right adjoint
$D : \mathbf{Bool} \to \mathbf{Moisil_n}$.

G. Georgescu, C. Vraciu, 1970

Our goal is to solve the equation

$$\frac{\textit{Game}}{\textit{Logic}} = \frac{\mathrm{x}}{(n+1)\text{-valued Moisil logic}}$$

joint work with Denisa Diaconescu

Let's take x = *Akinator* [http://en.akinator.com/]


akinator®
The Web Genie!
English
Play
Story of Akinator
Akinator on your smar
Submitted by starlord
I think of
Daenerys Targaryen
A Song of Ice and Fire

ish · Play · Story of Akinator · Akinator on your smartphone

If you find your character on the following list, please click on its name

Weasel (Deadpool)
Cross Bones (Marvel comics)
Baron zemo (Marvel)
Zola (scientist/Captain America)
Red Hulk (Red Hulk)
Obadiah Stane (Iron Man vilain)
The Winter Soldier (Bucky Barnes) (Captain America)

Akinator with n questions [the answer is displayed after n questions]

- The set of answers is a partially ordered set $(Ans, \prec)$:

$$Don't\ know \prec Probably \prec Yes \text{ and}$$
$$Don't\ know \prec Probably\ not \prec No.$$

- C is the set of characters, $\Delta \subseteq Ans^C$ is the set of data.
- We assume $\mathbf{1}, \mathbf{0} \in \Delta$.
- (question,answer): Q_δ^a with $\delta \in \Delta$ and $a \in Ans$.
- The knowledge after Q_δ^a is $K_\delta^a \in L_2^C$, where

  $K_\delta^a = \bigvee_{a \prec x} \chi_{\delta^{-1}(x)}$.

- A game is $\gamma := Q_{\delta_1}^{a_1} \cdot \cdots \cdot Q_{\delta_n}^{a_n}$.
- A state is $\sigma := Q_{\delta_1}^{a_1} \cdots Q_{\delta_k}^{a_k}$ with $k \leq n$; we denote $|\sigma| = k$.

Akinator with n questions

Interpretation $\mathcal{I}$

$$\begin{aligned}
\mathcal{I}_k(Q^a_\delta) &= (K^a_\delta, \ldots, K^a_\delta, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{k-1}), \text{ for } 1 \leq k \leq n \\
\mathcal{I}(Q^a_\delta) &= \mathcal{I}_1(Q^a_\delta), \\
\mathcal{I}(\sigma Q^a_\delta) &= \mathcal{I}(\sigma) \wedge \mathcal{I}_{k+1}(Q^a_\delta), \text{ where } |\sigma| = k.
\end{aligned}$$

Interpretation $\mathcal{V}$

$\gamma : C_n \subseteq \cdots \subseteq C_1 \subseteq C$

$\mathcal{I}(\gamma) = (\chi_{C_n}, \ldots, \chi_{C_1}) \subseteq (L_2^C)^{[n]}$, $\qquad \iota_C : (L_2^C)^{[n]} \xrightarrow{\sim} L^C_{n+1}$

$\mathcal{V}(\gamma) = \iota_C(\mathcal{I}(\gamma))$ the interpretation of γ in L^C_{n+1}

$\mathcal{V}(\gamma) : C \to L_{n+1}$

Akinator with n questions

Interpretation $\mathcal{V}$

$\gamma : C_n \subseteq \cdots \subseteq C_1 \subseteq C$
$\mathcal{I}(\gamma) = (\chi_{C_n}, \ldots, \chi_{C_1}) \subseteq (L_2^C)^{[n]}$, $\qquad \iota_C : (L_2^C)^{[n]} \xrightarrow{\sim} L_{n+1}^C$

$\mathcal{V}(\gamma) = \iota_C(\mathcal{I}(\gamma))$ the interpretation of γ in L_{n+1}^C
$\mathcal{V}(\gamma) : C \to L_{n+1}$

What is the meaning of $\mathcal{V}(\gamma)$?

Interpretation $\mathcal{V}$

$\gamma : C_n \subseteq \cdots \subseteq C_1 \subseteq C$

$\mathcal{I}(\gamma) = (\chi_{C_n}, \ldots, \chi_{C_1}) \subseteq (L_2^C)^{[n]}$, $\qquad \iota_C : (L_2^C)^{[n]} \xrightarrow{\sim} L_{n+1}^C$

$\mathcal{V}(\gamma) = \iota_C(\mathcal{I}(\gamma))$ the interpretation of γ in L_{n+1}^C

$\mathcal{V}(\gamma) : C \to L_{n+1}$

$$\mathcal{V}(\gamma)(c) = \tfrac{k}{n} \text{ iff } c \in C_k \setminus C_{k+1} \text{ iff}$$

the character c was eliminated from the selection by the $k+1$ question

Temporal interpretation of the Boolean nuances!

Akinator with n questions

We set:

- $G(C, \Delta, n) = \{\mathcal{V}(\gamma) \mid \gamma \text{ is a game}\} \subseteq L_{n+1}^{C}$
- $Ak(C, \Delta, n)$ be the $n+1$-valued Moisil algebra **generated** by $G(C, \Delta, n)$ in L_{n+1}^{C}.

Theorem

If α is a formula of the $n+1$-valued Moisil logic, the following are equivalent:

- $\vdash \alpha$,
- $e(\alpha) = 1$ for any C, Δ and $e : Form_{\mathcal{M}_{n+1}} \to Ak(C, \Delta, n)$ valuation.

$$\frac{\textit{Game}}{\textit{Logic}} \simeq \frac{\text{Akinator with } n \text{ questions}}{(n+1)\text{-valued Moisil logic}}$$

D.Diaconescu, I.L., FUZZ-IEEE 2017

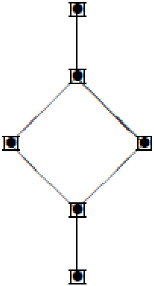
Nuances of truth

- Moisil logic is derived from the classical logic by the idea of nuancing, mathematically expressed by a categorical adjunction.
- Starting from a logical system and using the idea of nuance, it is possible to construct an n-nuanced logical system on the top of the given one.
- The *n-nuanced MV-algebras* are defined by nuancing Łukasiewicz logic, i.e. adding n nuances on top of an MV-algebra.

G. Georgescu, A. Popescu, 2006

Y|N Akinator with n questions and lies

2-nuanced MV_3-algebras



$$L_3^{[2]} = \{(0,0), (0,1/2), (0,1), (1/2,1/2), (1/2,1), (1,1)\}$$

A. Popescu, Tones of truth

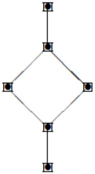
necessarily false, contingently false, contingently true, surprisingly false, true, as expected, and necessarily true

Y|N Akinator with n questions and 1 lie

$$\frac{\textit{Game}}{\textit{Logic}} \simeq \frac{\text{Y|N Akinator with } n \text{ questions and 1 lie}}{?}$$

Y|N Akinator with n questions and 1 lie

n-nuanced MV_3-algebras



$n = 2,\ L_3^{[2]}$

$$\frac{\textit{Game}}{\textit{Logic}} \simeq \frac{\text{Y|N Akinator with } n \text{ questions and 1 lie}}{\textbf{The theory of the } n\textbf{-nuanced MV}_3\textbf{-algebras}}$$

Y|N Akinator with n questions and 1 lie

$$\frac{\textit{Game}}{\textit{Logic}} \simeq \frac{\text{Y|N Akinator with } n \text{ questions and 1 lie}}{\textbf{The theory of the } n\textbf{-nuanced MV}_3\textbf{-algebras}}$$

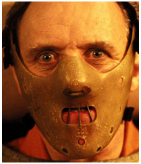
D.Diaconescu, I.L., Towards game semantics for nuanced logics, FUZZ-IEEE 2017, doi: 10.1109/FUZZ-IEEE.2017.8015600.

Classical Akinator with n questions (Yes/No)

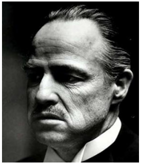
We use D. Diaconescu's slides from ManyVal 2017.

Classical Akinator with n questions

The Genie has a data base of characters:



Hannibal Lector



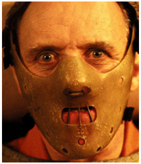
Don Vito Corleone

Princess Leia

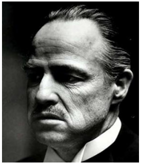
Jean-Luc Picard

Cruella de Vil

The Genie has a data base of characters:



Hannibal Lector



Don Vito Corleone

Princess Leia

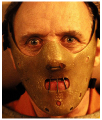
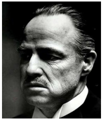
Jean-Luc Picard

Cruella de Vil

We have a set C of characters.

Classical Akinator with n questions

The Genie has a description of the characters:

					
villain	1	1	0	0	1
eccentric hair	0	0	1	1	1
bossy	1	1	1	1	1
european	1	1	0	1	1

The Genie asks questions of the form *"Is your character bossy?"*

The Genie's description of the characters is a set $\Delta \subseteq L_2^C$.

A question is an element $\delta \in \Delta$.

Classical Akinator with n questions

A round is a pair **(question,answer)**.

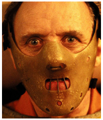
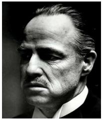
A round is represented as Q_δ^a where $\delta \in \Delta$ and $a \in L_2$.

- Is your character a villain? No.
- Q^0_{villain}

A game is a sequence of n rounds $\gamma := Q_{\delta_1}^{a_1} \dots Q_{\delta_n}^{a_n}$

Classical Akinator with n questions

The knowledge after the round Q^0_{villain} is

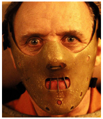
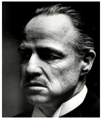
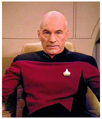
					
villain	1	1	0	0	1
K^0_{villain}	0	0	1	1	0

The knowledge after a round Q^a_δ is $K^a_\delta = \chi_{\delta^{-1}(a)} \in L_2^C$

where χ_Y is the characteristic function of Y in C.

Classical Akinator with n questions

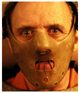
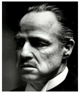
Suppose we have two rounds Q^1_{european} and Q^0_{villain}. The knowledge is

					
K^1_{european}	1	1	0	1	1
K^0_{villain}	0	0	1	1	0

After the two rounds, we choose the character

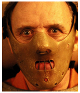
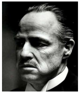
Classical Akinator with 3 questions

$$\gamma = Q^1_{\text{eccentric hair}} \; Q^0_{\text{villain}} \; Q^1_{\text{european}}$$

					
$K^1_{\text{eccentric hair}}$	0	0	1	1	1
K^0_{villain}	0	0	1	1	0
K^1_{european}	1	1	0	1	1

Classical Akinator with 3 questions

$\gamma = Q^1_{\text{eccentric hair}} \; Q^0_{\text{villain}} \; Q^1_{\text{european}}$

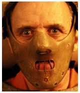
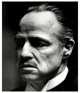
					
$K^1_{\text{eccentric hair}}$	0	0	1	1	1
K^0_{villain}	0	0	1	1	0
K^1_{european}	1	1	0	1	1

We have the following intermediate sets of characters

- $C_1 = \{$Princess Leia, Jean-Luc Picard, Cruella de Vil$\}$
- $C_2 = \{$Princess Leia, Jean-Luc Picard$\}$
- $C_3 = \{$Jean-Luc Picard$\}$

Classical Akinator with 3 questions

$\gamma = Q^1_{\text{eccentric hair}} \quad Q^0_{\text{villain}} \quad Q^1_{\text{european}}$

					
$K^1_{\text{eccentric hair}}$	0	0	1	1	1
K^0_{villain}	0	0	1	1	0
K^1_{european}	1	1	0	1	1

We have the following intermediate sets of characters

- $C_1 = \{$Princess Leia, Jean-Luc Picard, Cruella de Vil$\}$
- $C_2 = \{$Princess Leia, Jean-Luc Picard$\}$
- $C_3 = \{$Jean-Luc Picard$\}$

$\mathcal{V}(\gamma)(c) = \iota_C(\mathcal{I}(\gamma))(c) = \iota(\langle \chi_{C_1}(c), \chi_{C_2}(c), \chi_{C_3}(c)\rangle)$

- $\mathcal{V}(\gamma)(\text{Hannibal Lector}) = \iota(\langle 0, 0, 0\rangle) = 0$
- $\mathcal{V}(\gamma)(\text{Princess Leia}) = \iota(\langle 1, 1, 0\rangle) = \frac{2}{3}$
- $\mathcal{V}(\gamma)(\text{Jean-Luc Picard}) = \iota(\langle 1, 1, 1\rangle) = 1$